Brewster’s Clock Problem

Mahir Bilen Can

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1 Introduction

The claim below was observed and announced to the public by the clever bartender Brewster Pesses of the Sovereign Pub sometime after 8pm at the first Friday of the year 2014. Brewster came up with this problem when he was a child looking at the clock adding the digits and manipulating them. Here we propose a solution to his conjecture.

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1.1 Remainder of a number

For simplicity, let $n$ denote a positive number with at most 9 nonzero digits (these assumptions might be relaxed, if needed). For example, $n = 491572$. We write $n$ in the most general digit form, symbolically, as in

$$n = n_1 n_2 \ldots n_k,$$

so $n_1, \ldots, n_k$ are numbers from 1 to 9. For the above example, $n_1 = 4$, $n_2 = 9$, $n_3 = 1$, $n_4 = 5$, $n_5 = 7$ and $n_6 = 2$.

Define a new number $L(n)$ by adding the digits together: $L(n) = n_1 + n_2 + \cdots + n_k$. For example, $L(491572) = 4 + 9 + 1 + 5 + 7 + 2 = 28$. Note that if we apply the same operator $L$ to 28, we get 10, and applying it once more on 10, we obtain 1.

Observe that this is true in general: starting with $n$, if we compute $L(n)$, then $L(L(n))$, then $L(L(L(n)))$, we will eventually reduce to a single digit number. Let’s call this single digit number the remainder of $n$, and denote it by $B(n)$. 

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1.2 Brewster’s decomposition

Choose any subsequence of the sequence $n_1, \ldots, n_k$ of digits of $n = n_1 n_2 \cdots n_k$, say $n_{j_1}, \ldots, n_{j_r}$, and form a new (smaller) number $n(1)$ by concatenation:

$$n(1) := n_{j_1} n_{j_2} \cdots n_{j_r}.$$ 

Repeat this construction on the number obtained from $n$ by removing the digits of $n_{j_1} n_{j_2} \cdots n_{j_r}$, call the newly created, smaller number by $n(2)$. Repeat until no number is left from $n$.

For example, for $n = 491572$, first choose $n_{j_1} = 1, n_{j_2} = 7$ to form 17, then choose $n_{i_1} = 9, n_{i_2} = 5, n_{i_3} = 2$ to form 952, and finally, choose $n_{s_1} = 4$ to form 4. Thus, $n(1) = 17, n(2) = 952$ and $n(3) = 4$.

For our purposes, the order which we form the smaller numbers is not important. Instead of 17, we could just write 71. However, for simplicity we keep the original ordering of the digits in $n$. The important point is to form these smaller numbers by consuming all digits of $n$.

1.3 The claim

When we append two numbers $a$ and $b$ to get a larger number $n$ we prefer to write it as $n = a \ast b$ (this is not a multiplication, but just the concatenation of $a$ and $b$). For example, if $n = 5379422$, then we write $n = 537 \ast 9422$. (If this is confusing, just forget the $\ast$.)

Suppose we have the smaller numbers $n(1), \ldots, n(r)$ as a result of the above described decomposition of $n$. The claim is that if we form a new number by concatenating the remainders $B(n(1)), \ldots, B(n(r))$ of $n(1), n(2), \ldots, n(r)$’s, and then compute the remainder of the resulting number, we obtain the remainder of $n$. In mathematical terms, Brewster’s Claim is the following equality:

$$B(B(n(1)) \ast B(n(2)) \cdots \ast B(n(r))) = B(n) \quad (1.1)$$

For our example from the previous subsection, the remainders of $n(1) = 17, n(2) = 952$ and $n(3) = 4$, are respectively, 8, 7 and 4. The new number we form is 874, and its remainder is 1, which is equal to $B(n) = B(491572)$. 

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Remark 1.2. Observe that to prove the claim (1.1), it is enough to prove it for decompositions with two parts only:

\[ B(n) = B(B(n(1)) \ast B(n(2))). \]  

(1.3)

The proof of the general case follows from this by induction.

2 Proof

The proof will follow from some simple observations on the operator \( L \). It follows from the ‘at most 10 digits’ assumption that \( L(n) = n_1 + \cdots + n_9 \leq 81 \) since each \( n_i \leq 9 \) for \( i = 1, \ldots, 9 \). In other words, \( L(n) \) has at most 2 digits. In particular, \( L(L(n)) \) has at most two digits: \( L^2(n) = a \ast b \) with \( a \leq 1, b \leq 8 \). Consequently, we see that for such \( n \), \( L^3(n) = L(L(L(n))) \) is already the remainder \( B(n) \) of \( n \).

Another easy observation regarding the operator \( L \) is the following:

Lemma 2.1. If \( n \) is a concatenation \( n = a \ast b \) of two numbers \( a \) and \( b \) (for example \( a = 49, b = 1572 \) for \( n = 491572 \)), then \( L(n) = L(a) + L(b) \).

Proof. Suppose the digits of \( a \) and \( b \) are in \( a = a_1 \cdots a_p, b = b_1 \cdots b_q \). Then \( a \ast b = a_1 \cdots a_p b_1 \cdots b_q \). Then \( L(n) = L(a \ast b) = a_1 + \cdots + a_p + b_1 + \cdots b_q = L(a) + L(b) \). \( \square \)

Suppose we have \( n = a \ast b \) as in the proof of Lemma 2.1. Notice that, since \( L(a) \) and \( L(b) \) both have at most two digits, say \( L(a) = x_1 \ast x_2 \), and \( L(b) = y_1 \ast y_2 \), we have

\[ B(n) = L^3(n) = L^2(L(n)) = L^2(L(a) + L(b))) = L^2(x_1 \ast x_2 + y_1 \ast y_2). \]

Recall that the remainder of division by 10 is called \( \mod 10 \). Recall also the basic modular arithmetic:

\[ (x \mod 10) + (y \mod 10) = x + y \mod 10. \]  

(2.2)

In the light of the trivial identity \( X = (X \mod 10) + X - (X \mod 10) \), and the fact that \( x_1 \ast x_2 + y_1 \ast y_2 \leq 81 \), we re-write \( L(n) \) as follows: \( L(n) = x_1 \ast x_2 + y_1 \ast y_2 \mod 10 \) \( L(n) = (x_1 \ast x_2 + y_1 \ast y_2 \mod 10) + x_1 \ast x_2 + y_1 \ast y_2 - ((x_1 \ast x_2 + y_1 \ast y_2) \mod 10) \).

The following lemma is elementary, and there is no need for a proof:

Lemma 2.3. For a two digit number \( x = v \ast u \), we have \( u = (x \mod 10) \) and \( v = (x - (x \mod 10)) / 10 \). Thus \( L(x) = L(v \ast u) = v + u \).
Therefore, by Lemma 2.3, $L^2(n) = L(x_1 \times x_2 + y_1 \times y_2)$ is equal to $(x_1 \times x_2 + y_1 \times y_2 \mod 10) + \frac{x_1 \times x_2 + y_1 \times y_2 - ((x_1 \times x_2 + y_1 \times y_2 \mod 10))}{10}$ = $(x_1 \times x_2 + y_1 \times y_2 \mod 10) + \frac{y_1 \times y_2 - (y_1 \times y_2 \mod 10)}{10}$ = $[(x_1 \times x_2 \mod 10) + \frac{y_1 \times y_2 - (y_1 \times y_2 \mod 10)}{10}] = L(x_1 \times x_2) + L(y_1 \times y_2)$.

In other words,

$$L^2(n) = L(x_1 \times x_2) + L(y_1 \times y_2) = L^2(a) + L^2(b). \quad (2.4)$$

Proof of Brewster’s Claim. Since $L(x_1 \times x_2)$ and $L(y_1 \times y_2)$ have at most two digits, by iterating (2.4), we see that $L^i(x_1 \times x_2 + y_1 \times y_2) = L^i(x_1 \times x_2) + L^i(y_1 \times y_2)$ for any $i \geq 1$. Since $B(n) = L^i(n)$ for any $i \geq 3$, we might as well look at $L^6(n) = B(n)$:

$$B(n) = L^6(n) = L^5(L(n)) = L^2(L^3(a) + L^3(b)) = L^2(B(a) + B(b)) = L^3(B(a) \times B(b)) = B(B(a) \times B(b)).$$

$\square$