Integral Transforms and Special Functions

Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gitr20

Identities for generalized Euler polynomials

Lin Jiu\(^{a}\), Victor H. Moll\(^{a}\) & Christophe Vignat\(^{b}\)
\(^{a}\) Department of Mathematics, Tulane University, New Orleans, LA 70118, USA
\(^{b}\) LSS-Supelec, Université Orsay Paris Sud 11, France
Published online: 06 Jun 2014.

To cite this article: Lin Jiu, Victor H. Moll & Christophe Vignat (2014) Identities for generalized Euler polynomials, Integral Transforms and Special Functions, 25:10, 777-789, DOI: 10.1080/10652469.2014.918613

To link to this article: http://dx.doi.org/10.1080/10652469.2014.918613

PLEASE SCROLL DOWN FOR ARTICLE

Taylor & Francis makes every effort to ensure the accuracy of all the information (the “Content”) contained in the publications on our platform. However, Taylor & Francis, our agents, and our licensors make no representations or warranties whatsoever as to the accuracy, completeness, or suitability for any purpose of the Content. Any opinions and views expressed in this publication are the opinions and views of the authors, and are not the views of or endorsed by Taylor & Francis. The accuracy of the Content should not be relied upon and should be independently verified with primary sources of information. Taylor and Francis shall not be liable for any losses, actions, claims, proceedings, demands, costs, expenses, damages, and other liabilities whatsoever or howsoever caused arising directly or indirectly in connection with, in relation to or arising out of the use of the Content.

This article may be used for research, teaching, and private study purposes. Any substantial or systematic reproduction, redistribution, reselling, loan, sub-licensing, systematic supply, or distribution in any form to anyone is expressly forbidden. Terms &
Conditions of access and use can be found at http://www.tandfonline.com/page/terms-and-conditions
Identities for generalized Euler polynomials

Lin Jiua, Victor H. Moll* and Christophe Vignatb

aDepartment of Mathematics, Tulane University, New Orleans, LA 70118, USA; bLSS-Supelec, Université Orsay Paris Sud 11, France

(Received 31 January 2014; accepted 23 April 2014)

For \(N \in \mathbb{N}\), let \(T_N\) be the Chebyshev polynomial of the first kind. Expressions for the sequence of numbers \(p(\ell;N)\), defined as the coefficients in the expansion of \(1/T_N(1/z)\), are provided. These coefficients give formulas for the classical Euler polynomials in terms of the so-called generalized Euler polynomials. The proofs are based on a probabilistic interpretation of the generalized Euler polynomials recently given by Klebanov et al. Asymptotics of \(p(\ell;N)\) are also provided.

**Keywords:** generalized Euler polynomials; hyperbolic secant distributions; Chebyshev polynomials

**2010 Mathematics Subject Classification:** Primary: 11B68; Secondary: 60E05

1. Introduction

The Euler numbers \(E_n\), defined by the generating function

\[
\frac{1}{\cosh z} = \sum_{n=0}^{\infty} E_n \frac{z^n}{n!}
\]

(1.1)

and the Euler polynomials \(E_n(x)\) that generalize them

\[
\sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} = \frac{2e^{xz}}{e^z + 1}
\]

(1.2)

\([1, 9.630, 9.651]\) are examples of basic special functions. It follows directly from the definition that \(E_n = 0\) for \(n\) odd. Moreover, the relation \(E_n = 2^n E_n(\frac{1}{2})\) follows by setting \(x = \frac{1}{2}\) in (1.2), replacing \(z\) by \(2z\) and comparing with (1.1).

Moreover, the identity

\[
\frac{2e^{xz}}{e^z + 1} = \frac{2e^{(x-1/2)z}}{e^{z/2} + e^{-z/2}}
\]

(1.3)

produces

\[
E_n(x) = \sum_{k=0}^{n} \binom{n}{k} E_k \left( x - \frac{1}{2} \right)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} E_k \left( \frac{1}{2} \right)^{n-k} \left( x - \frac{1}{2} \right)^{n-k},
\]

(1.4)

that gives \(E_n(x)\) in terms of the Euler numbers \([1, 9.650]\).

*Corresponding author. Emails: vhm@tulane.edu; vhm@math.tulane.edu

© 2014 Taylor & Francis
The generalized Euler polynomials $E_n^{(p)}(z)$, defined by the generating function

$$
\sum_{n=0}^{\infty} E_n^{(p)}(x) \frac{z^n}{n!} = \left( \frac{2}{1 + e^z} \right)^p e^{xz} \quad \text{for } p \in \mathbb{N}
$$

are polynomials extending $E_n(x)$, the case $p = 1$. These appear in [2, Section 24.16]. The definition leads directly to the expression

$$
E_n^{(p)}(x) = \sum_{k=0}^{n} \binom{n}{k} x^k E_n^{(p-1)}(0),
$$

where the generalized Euler numbers $E_n^{(p)}(0)$ are defined recursively by

$$
E_n^{(p)}(0) = \sum_{k=0}^{n} \binom{n}{k} E_n^{(p-1)}(0) E_{n-k}(0),
$$

for $p > 1$ and initial condition $E_n^{(1)}(0) = E_n(0)$.

# 2. A probabilistic representation of Euler polynomials and their generalizations

This section discusses probabilistic representations of the Euler polynomials and their generalizations. The results involve the expectation operator $\mathbb{E}$ defined by

$$
\mathbb{E}g(L) = \int g(x)f_L(x) \, dx,
$$

with $f_L(x)$ being the probability density of the random variable $L$ and for any function $g$ such that the integral exists.

**Proposition 2.1** Let $L$ be a random variable with hyperbolic secant density

$$
f_L(x) = \text{sech } \pi x \quad \text{for } x \in \mathbb{R}.
$$

Then the Euler polynomial is given by

$$
E_n(x) = \mathbb{E}(x + iL - \frac{1}{2})^n, \quad i^2 = -1.
$$

**Proof** The right-hand side of (2.3) is

$$
\mathbb{E} \left( x + iL - \frac{1}{2} \right)^n = \int_{-\infty}^{\infty} \left( x - \frac{1}{2} + it \right)^n \text{sech } \pi t \, dt
$$

$$
= \sum_{j=0}^{n} \binom{n}{j} \left( x - \frac{1}{2} \right)^{n-j} i^j \int_{-\infty}^{\infty} t^j \text{sech } \pi t \, dt.
$$

The identity

$$
\int_{-\infty}^{\infty} t^k \text{sech } \pi t \, dt = \frac{|E_k|}{2^k}
$$

(2.4)
holds for \( k \) odd, since both sides vanish and for \( k \) even, it appears as entry 3.523.4 in [1]. A proof of this entry may be found in [3]. Then, using \(|E_{2n}| = (-1)^n E_{2n}\) (entry 9.633 in [1])

\[
\mathbb{E} \left( x + iL - \frac{1}{2} \right)^n = \sum_{j=0}^{n} \binom{n}{j} \left( x - \frac{1}{2} \right)^{n-j} \frac{E_{j}}{2^{j}} = E_n(x). \tag{2.5}
\]

There is a natural extension to the case of \( E_{n}^{(p)}(x) \): it requires choosing \( p \) independent random variables \( L_1, L_2, \ldots, L_p \), all having the hyperbolic secant distribution (2.2).

**Theorem 2.2**  
Let \( p \in \mathbb{N} \) and \( \{L_j : 1 \leq j \leq p\} \) be a collection of independent identically distributed random variables with hyperbolic secant distribution. Then

\[
E_n^{(p)}(x) = \mathbb{E} \left[ x + \sum_{j=1}^{p} \left( iL_j - \frac{1}{2} \right) \right]^n. \tag{2.6}
\]

The proof is similar to the previous case, so it is omitted.

In a recent paper, Klebanov et al. [4] considered random sums of independent random variables of the form

\[
\frac{1}{N} \sum_{j=1}^{\mu_N} L_j, \tag{2.7}
\]

where the random number of summands \( \mu_N \) is independent of the \( L_j \)'s and is described below.

**Definition 2.3**  
Let \( N \in \mathbb{N} \) and \( T_N(z) \) be the Chebyshev polynomial of the first kind. The random variable \( \mu_N \) taking values in \( \mathbb{N} \), is defined by its generating function

\[
\mathbb{E}z^{\mu_N} = \frac{1}{T_N(1/z)}. \tag{2.8}
\]

Information about the Chebyshev polynomials appears in [1,2].

**Example 2.4**  
Take \( N = 2 \). Then \( T_2(z) = 2z^2 - 1 \) gives

\[
\mathbb{E}z^{\mu_2} = \frac{1}{T_2(1/z)} = \frac{z^2}{2 - z^2} = \sum_{\ell=1}^{\infty} \frac{z^{2\ell}}{2\ell}. \tag{2.9}
\]

Therefore \( \mu_2 \) takes the value \( 2\ell \), with \( \ell \in \mathbb{N} \), with probability

\[
\Pr(\mu_2 = 2\ell) = 2^{-\ell}. \tag{2.10}
\]

In [4], Klebanov et al. prove the following result.

**Theorem 2.5** (Klebanov et al.)  
Assume \( \{L_j\} \) is a sequence of independent identically distributed random variables with hyperbolic secant distribution. Then, for all \( N \geq 2 \) and \( \mu_N \) defined in (2.8), the random variable

\[
L := \frac{1}{N} \sum_{j=1}^{\mu_N} L_j \tag{2.11}
\]

has the same hyperbolic secant distribution.
3. The Euler polynomials in terms of the generalized ones

The identities 1.6 and 1.7 can be used to express the generalized Euler polynomial \( E_n^{(p)}(x) \) in terms of the standard Euler polynomials \( E_n(x) \). However, to the best of our knowledge, there is no formula in the literature that expresses \( E_n(x) \) in terms of \( E_n^{(p)}(x) \). This section presents such a formula.

**Definition 3.1** Let \( N \in \mathbb{N} \). The sequence \( \{ p_{\ell}^{(N)} \} : \ell = 0, 1, \ldots \) is defined as the coefficients in the expansion

\[
\frac{1}{T_N(1/z)} = \sum_{\ell=0}^{\infty} p_{\ell}^{(N)} z^\ell. \tag{3.1}
\]

Definition 2.3 shows that

\[
p_{\ell}^{(N)} = \Pr(\mu_N = \ell) \text{ for } \ell \in \mathbb{N}. \tag{3.2}
\]

The numbers \( p_{\ell}^{(N)} \) will be referred as the probability numbers.

**Example 3.2** For \( N = 2 \), Example 2.4 gives

\[
p_{\ell}^{(2)} = \begin{cases} 
0 & \text{if } \ell \text{ is odd,} \\
2^{-\ell/2} & \text{if } \ell \text{ is even, } \ell \neq 0. 
\end{cases} \tag{3.3}
\]

The coefficients \( p_{\ell}^{(N)} \) are now used to produce expansions of \( E_n(x) \), one for each \( N \in \mathbb{N} \), in terms of the generalized Euler polynomials.

**Theorem 3.3** The Euler polynomials satisfy, for all \( N \in \mathbb{N} \),

\[
E_n(x) = \frac{1}{N^n} \mathbb{E} \left[ E_n^{(\mu_N)} \left( \frac{1}{2} \mu_N + N \left( x - \frac{1}{2} \right) \right) \right]. \tag{3.4}
\]

**Proof** From (2.3) and (2.11)

\[
E_n \left( \frac{1}{2} \right) = \mathbb{E}(tL)^n = \frac{1}{N^n} \mathbb{E} \left[ t \sum_{j=1}^{\mu_N} L_j \right]^n, \tag{3.5}
\]

with Theorem 2.2, this yields

\[
\mathbb{E} \left[ E_n^{(\mu_N)} \left( \frac{\mu_N}{2} \right) \right] = \mathbb{E} \left[ t \sum_{j=1}^{\mu_N} L_j \right]^n = N^n E_n \left( \frac{1}{2} \right). \tag{3.6}
\]
Using identity 1.4, it follows that

\[ E_n(x) = \sum_{k=0}^{n} \binom{n}{k} E_k \left( \frac{1}{2} \right) \left( x - \frac{1}{2} \right)^{n-k} \]

\[ = \mathbb{E} \left[ \sum_{k=0}^{n} \binom{n}{k} N^{-k} E_k^{(\mu_N)} \left( \frac{1}{2} \mu_N \right) \left( x - \frac{1}{2} \right)^{n-k} \right] \]

\[ = \mathbb{E} \left[ \sum_{k=0}^{n} \binom{n}{k} N^{-k} (\ell L_1 + \cdots + \ell L_{\mu_N})^k \left( x - \frac{1}{2} \right)^{n-k} \right] \]

\[ = \mathbb{E} \left[ \frac{1}{N^n} \sum_{k=0}^{n} \binom{n}{k} (\ell L_1 + \cdots + \ell L_{\mu_N})^k \left( N \left( x - \frac{1}{2} \right) \right)^{n-k} \right] \]

\[ = \mathbb{E} \left[ \frac{1}{N^n} \left( \ell L_1 + \cdots + \ell L_{\mu_N} + N \left( x - \frac{1}{2} \right) \right)^n \right] \]

\[ = \frac{1}{N^n} \mathbb{E} \left[ E_n^{(\mu_N)}(z) \right], \]

where \( z = \frac{1}{2} \mu_N + N(x - \frac{1}{2}) \). This completes the proof.

The next result is established using the fact that the expectation operator \( \mathbb{E} \) satisfies

\[ \mathbb{E} [h(\mu_N)] = \sum_{k=0}^{\infty} p_k^{(N)} h(k) \quad (3.7) \]

for any function \( h \) such that the right-hand side exists.

**Corollary 3.4** The Euler polynomials satisfy

\[ E_n(x) = \frac{1}{N^n} \sum_{k=N}^{\infty} p_k^{(N)} E_n^{(k)} \left( \frac{1}{2} k + N \left( x - \frac{1}{2} \right) \right). \quad (3.8) \]

**Note 3.5** Corollary 3.4 gives an infinite family of expressions for \( E_n(x) \) in terms of the generalized Euler polynomials \( E_n^{(k)}(x) \), one for each value of \( N \geq 2 \).

**Example 3.6** The expansion (3.8) with \( N = 2 \) gives

\[ E_n(x) = \frac{1}{2^n} \sum_{\ell=1}^{\infty} \frac{1}{2^\ell} E_n^{(2\ell)}(\ell + 2x - 1). \quad (3.9) \]

For instance, when \( n = 1 \),

\[ E_1(x) = \frac{1}{2} \sum_{\ell=1}^{\infty} \frac{1}{2^\ell} E_1^{(2\ell)}(\ell + 2x - 1) \quad (3.10) \]
and the value \( E_1^{(\ell)}(x) = x - \ell/2 \) gives

\[
E_1(x) = \frac{1}{2} \sum_{\ell=1}^{\infty} \frac{1}{2^\ell} (\ell + 2x - 1 - \ell) = x - \frac{1}{2}
\]

as expected.

4. The probability numbers

For fixed \( N \in \mathbb{N} \), the random variable \( \mu_N \) has been defined by its moment-generating function

\[
\mathbb{E}z^{\mu_N} = \frac{1}{T_N(1/z)} = \sum_{\ell=0}^{\infty} p^{(N)}_\ell z^\ell.
\]

This section presents properties of the probability numbers \( p^{(N)}_\ell \) that appear in Corollary 3.4.

For small \( N \), the coefficients \( p^{(N)}_\ell \) can be computed directly by expanding the rational function \( 1/T_N(1/z) \) in partial fractions. Example 2.4 gave the case \( N = 2 \). The cases \( N = 3 \) and \( N = 4 \) are presented below.

Example 4.1 For \( N = 3 \), the Chebyshev polynomial is

\[
T_3(z) = 4z^3 - 3z = 4z(z - \alpha)(z + \alpha),
\]

with \( \alpha = \sqrt{3}/2 \). This yields

\[
\frac{1}{T_3(1/z)} = \frac{z^3}{4(1 - \alpha z)(1 + \alpha z)} = \sum_{k=0}^{\infty} \frac{3^k}{2^{2k+2}} z^{2k+3}.
\]

It follows that \( p^{(3)}_\ell = 0 \) unless \( \ell = 2k + 3 \) and

\[
p^{(3)}_{2k+3} = \frac{3^k}{2^{2k+2}}.
\]

Corollary 3.4 now gives

\[
E_n(x) = \frac{1}{3^n} \sum_{k=0}^{\infty} \frac{3^k}{2^{2k+2}} E_n^{(2k+3)}(3x + k),
\]

a companion to (3.9).

Example 4.2 The probability numbers for \( N = 4 \) are computed from the expression

\[
\frac{1}{T_4(1/z)} = \frac{1}{z^4 - 8z^2 + 8}.
\]

The factorization

\[
z^4 - 8z^2 + 8 = (z^2 - \beta)(z^2 - \gamma)
\]

Downloaded by [Tulane University] at 12:35 19 August 2015
with $\beta = 2(2 + \sqrt{2})$ and $\gamma = 2(2 - \sqrt{2})$ and the partial fraction decomposition

$$\frac{z^4}{z^4 - 8z^2 + 8} = \frac{\beta}{\beta - \gamma} \frac{1}{1 - \beta/z^2} - \frac{\gamma}{\beta - \gamma} \frac{1}{1 - \gamma/z^2}$$

(4.8)

show that $p^{(4)}_\ell = 0$ for $\ell$ odd or $\ell = 2$ and

$$p^{(4)}_{2\ell} = \frac{\sqrt{2}}{2^{2\ell+1}}[(2 + \sqrt{2})^{\ell-1} - (2 - \sqrt{2})^{\ell-1}]$$

(4.9)

for $\ell \geq 2$. Corollary 3.4 now gives

$$E_n(x) = \sqrt{2} \sum_{\ell=2}^{\infty} \frac{[(2 + \sqrt{2})^{\ell-1} - (2 - \sqrt{2})^{\ell-1}]}{2^{2\ell+1}} E_n^{(2\ell)}(4x + \ell - 2).$$

(4.10)

Some elementary properties of the probability numbers are presented next.

**Proposition 4.3** The probability numbers $p^{(N)}_\ell$ vanish if $\ell < N$.

*Proof* The Chebyshev polynomial $T_N(z)$ has the form $2^{N-1}z^N + \text{lower order terms}$. Then the expansion of $1/T_N(1/z)$ has a zero of order $N$ at $z = 0$. This proves the statement.

**Proposition 4.4** The probability numbers $p^{(N)}_\ell$ vanish if $\ell \not\equiv N \mod 2$.

*Proof* The polynomial $T_N(z)$ has the same parity as $N$. The same holds for the rational function $1/T_N(1/z)$.

An expression for the probability numbers is given next.

**Theorem 4.5** Let $N \in \mathbb{N}$ be fixed and define

$$\theta_k^{(N)} = \frac{(2k - 1)\pi}{2N}.$$  

(4.11)

Then

$$p^{(N)}_\ell = \frac{1}{N} \sum_{k=1}^{N} (-1)^{k+1} \sin \theta_k^{(N)} \cos^{\ell-1} \theta_k^{(N)}.$$  

(4.12)

Since, as shown in [4], these $p^{(N)}_\ell$ are probability numbers, they satisfy $0 \leq p^{(N)}_\ell \leq 1$, a property that does not appear immediately from these finite sums expressions.
Proof  The Chebyshev polynomial is defined by \( T_N(\cos \theta) = \cos(N\theta) \), so its roots are \( z_k^{(N)} = \cos \theta_k^{(N)} \), with \( \theta_k^{(N)} \) as above. The leading coefficient of \( T_N(z) \) is \( 2^{N-1} \), thus

\[
\frac{1}{T_N(z)} = \frac{2^{1-N}}{\prod_{k=1}^{N}(z - z_k)}. \tag{4.13}
\]

In the remainder of the proof, the superscript \( N \) has been dropped from \( z_k^{(N)} \) and \( \theta_k^{(N)} \), for clarity. Define

\[
Q(z) = \prod_{k=1}^{N}(z - z_k). \tag{4.14}
\]

The roots \( z_k \) of \( Q \) are distinct, therefore

\[
\frac{1}{Q(z)} = \sum_{k=1}^{N} \frac{1}{Q(z_k)} \frac{1}{z - z_k}. \tag{4.15}
\]

The identity \( T_N(z) = NU_{N-1}(z) \) gives

\[
Q'(z_k) = N2^{1-N}U_{N-1}(z_k), \tag{4.16}
\]

where \( U_j(z) \) is the Chebyshev polynomial of the second kind defined by

\[
U_N(\cos \theta) = \frac{\sin(N+1)\theta}{\sin \theta}. \tag{4.17}
\]

Then

\[
U_{N-1}(z_k) = U_{N-1}(\cos \theta_k) = \frac{\sin N\theta_k}{\sin \theta_k}, \tag{4.18}
\]

and the value \( \sin N\theta_k = (-1)^{k+1} \) yields

\[
Q'(z_k) = \frac{(-1)^{k+1}}{\sin \theta_k}N2^{1-N}. \tag{4.19}
\]

Therefore (4.15) now gives

\[
\frac{1}{Q(z)} = \frac{2^{N-1}}{N} \sum_{k=1}^{N} \frac{(-1)^{k+1} \sin \theta_k}{z - \cos \theta_k}. \tag{4.20}
\]

It follows that

\[
\frac{1}{T_N(1/z)} = \frac{Q(1/z)}{Q(1/z)} = \frac{1}{N} \sum_{k=1}^{N} (-1)^{k+1} \frac{z \sin \theta_k}{1 - z \cos \theta_k}
\]

\[
= \frac{1}{N} \sum_{k=1}^{N} (-1)^{k+1} \sin \theta_k \sum_{\ell=0}^{\infty} z^{\ell+1} \cos^{\ell} \theta_k
\]

\[
= \frac{1}{N} \sum_{\ell=0}^{\infty} z^{\ell+1} \sum_{k=1}^{N} (-1)^{k+1} \sin \theta_k \cos^{\ell} \theta_k.
\]

The proof is complete. ■
The next result provides another explicit formula for the probability numbers. The coefficients $A(n, k)$ appear in OEIS entry A008315, as entries of the Catalan triangle.

**Theorem 4.6** Let $A(n, k) = \binom{n}{k} - \binom{n}{k-1}$. Then, if $N \equiv \ell \mod 2$,

$$p^{(N)}_{\ell} = \frac{1}{2\ell} \sum_{t=\lfloor (1/2)((\ell/N) - 1) \rfloor}^{\lfloor (1/2)((\ell/N) - 1) \rfloor} (-1)^t A\left(\ell - 1, \frac{1}{2}(\ell - (2t + 1)N)\right),$$

when $\ell$ is not an odd multiple of $N$ and

$$p^{(N)}_{\ell} = \frac{1}{2\ell} \sum_{s=1}^{\lfloor \ell/N - 1 \rfloor} (-1)^{k-s} A(\ell - 1, sN) + \frac{(-1)^k}{2^{\ell-1}} \text{ with } k = \frac{1}{2} \left( \frac{\ell}{N} - 1 \right)$$

otherwise.

The proof begins with a preliminary result.

**Lemma 4.7** Let $N \in \mathbb{N}$ and $\theta_k = (\pi/2)((2k - 1)/N)$. Then

$$f_N(z) = \sum_{k=1}^{N} (-1)^{k+1} e^{i\theta_k z}$$

is given by

$$f_N(z) = 1 - (-1)^N e^{\pi iz} \quad \text{if } z \neq (2t + 1)N \quad \text{with } t \in \mathbb{Z}$$

and

$$f_N(z) = (-1)^{\ell} N \ell \quad \text{if } z = (2t + 1)N \text{ for some } t \in \mathbb{Z}.$$ 

In particular

$$f_N(k) = \begin{cases} (-1)^{(k/N-1)/2} N \ell & \text{if } k/N \text{ is an odd integer} \\ 1 - (-1)^{N+k} \quad 2 \cos(\pi k/2N) & \text{otherwise.} \end{cases}$$

**Proof** The function $f_N$ is the sum of a geometric progression. The formula (4.23) comes from (4.22) by passing to the limit.

The proof of Theorem 4.6 is given now.
Proof  The expression for \( p^{(N)}_\ell \) given in Theorem 4.5 yields

\[
p^{(N)}_\ell = \frac{1}{N} \sum_{k=1}^{N} (-1)^{k+1} \left( \frac{e^{i\theta_k} - e^{-i\theta_k}}{2i} \right) \left( \frac{e^{i\theta_k} + e^{-i\theta_k}}{2} \right)^{\ell-1}
\]

\[
= \frac{1}{2^{\ell} Ni} \sum_{k=1}^{N} (-1)^{k+1} \sum_{r=0}^{\ell-1} \binom{\ell-1}{r} [f_N(l - 2r) - f_N(l - 2r - 2)]
\]

\[
= \frac{1}{2^{\ell} Ni} \sum_{r=1}^{\ell-1} \left[ \sum_{r=1}^{\ell-1} A(\ell - 1, r) f_N(\ell - 2r) + f_N(\ell) - f_N(\ell) \right].
\]

Now \( f_N(\ell) = f_N(-\ell) = 0 \) if \( \ell/N \) is not an odd integer. On the other hand, if \( \ell = (2t + 1)N \), with \( t \in \mathbb{Z} \), then

\[
f_N(\ell) = (-1)^{t} N t \quad \text{and} \quad f_N(-\ell) = (-1)^{t} N t.
\]

Thus

\[
f_N(\ell) - f_N(-\ell) = \begin{cases} 2N t (-1)^{(\ell/N - 1)/2} & \text{if } \ell \text{ is an odd multiple of } N, \\ 0 & \text{otherwise}. \end{cases}
\]

The simplification of the previous expression for \( p^{(N)}_\ell \) is divided in two cases, according to whether \( \ell \) is an odd multiple of \( N \) or not.

Case 1  Assume \( \ell \) is not an odd multiple of \( N \). Then

\[
p^{(N)}_\ell = \frac{1}{2^{\ell} Ni} \sum_{r=0}^{\ell-1} A(\ell - 1, r) f_N(\ell - 2r).
\]

Moreover,

\[
f_N(\ell - 2r) = \begin{cases} (-1)^{t} N t & \text{if } \frac{l - 2r}{N} = 2t + 1, \\ 0 & \text{otherwise}. \end{cases}
\]

Therefore

\[
p^{(N)}_\ell = \frac{1}{2^{\ell}} \sum_{r=(1/2)((\ell/N - 1)/\ell - 2r = (2t + 1)N}^{(1/2)((\ell/N - 1)/\ell)} (-1)^{t} A(\ell - 1, r).
\]

Observe that \( \ell - (2t + 1)N \) is always an even integer, thus the index \( r \) may be eliminated from the previous expression to obtain

\[
p^{(N)}_\ell = \frac{1}{2^{\ell}} \sum_{r=[(1/2)((\ell/N - 1)]}^{(1/2)((\ell/N - 1])} (-1)^{t} A \left( \ell - 1, \frac{1}{2}(\ell - (2t + 1)N) \right).
\]
Case 2 Assume $\ell$ is an odd multiple of $N$, say $\ell = (2k + 1)N$. Then

$$p_{\ell}^{(N)} = \frac{1}{2^\ell Ni} \left[ \sum_{r=0}^{\ell-1} A(\ell - 1, r) f_{N}(\ell - 2r) + 2Ni(-1)^k \right]$$

$$= \frac{1}{2^\ell Ni} \left[ \sum_{r=0}^{\ell-1} A(\ell - 1, r) f_{N}(\ell - 2r) \right] + \frac{(-1)^k}{2^{\ell-1}}.$$

The term $f_{N}(\ell - 2r)$ vanishes unless $\ell - 2r$ is an odd multiple of $N$. Given that $\ell = (2k + 1)N$, the term is non-zero provided $2r$ is an even multiple of $N$; say $r = sN$ for $s \in \mathbb{N}$. The range of $s$ is $1 \leq s \leq (\ell - 1)/N = 2k + 1 - 1/N$. This implies $1 \leq s \leq 2k = \ell/N - 1$, and it follows that

$$p_{\ell}^{(N)} = \frac{1}{2^\ell} \sum_{s=1}^{\ell/N-1} (-1)^{k-s} A(\ell - 1, sN) + \frac{(-1)^k}{2^{\ell-1}}, \quad \text{with } k = \frac{1}{2} \left( \frac{\ell}{N} - 1 \right).$$

The proof is complete.

Note 4.8 The expression in Theorem 4.6 shows that $p_{\ell}^{(N)}$ is a rational number with a denominator a power of 2 of exponent at most $\ell$. Arithmetic properties of these coefficients will be described in a future publication [5]. Moreover, the probability numbers $p_{\ell}^{(N)}$ appear in the description of a random walk on $N$ sites. Details will appear in [5].

5. An asymptotic expansion

The final result deals with the asymptotic behaviour of the probability numbers $p_{\ell}^{(N)}$.

THEOREM 5.1 Let $\varphi_N(z) = \mathbb{E}[z^{\mu_N}]$. Then, for fixed $z$ in the unit disk $|z| < 1$,

$$\varphi_N(z) \sim \left( \frac{z}{1 + \sqrt{1-z^2}} \right)^N, \quad \text{as } N \to \infty. \quad (5.1)$$

Proof The generating function satisfies

$$\varphi_N(z) = \frac{1}{T_N(1/z)} = \frac{z^N}{2^{N-1}} \prod_{k=1}^{N} (1 - z \cos \theta_k^{(N)})^{-1} \quad (5.2)$$

with $\theta_k^{(N)} = (2k - 1)\pi/2N$ as before. Then

$$\log \varphi_N(z) = \log 2 + N \log \frac{z}{2} - \sum_{k=1}^{N} \log(1 - z \cos \theta_k^{(N)}). \quad (5.3)$$

For large $N$, the last sum is approximated by a Riemann integral

$$\frac{1}{N} \sum_{k=1}^{N} \log(1 - z \cos \theta_k^{(N)}) \sim \frac{1}{\pi} \int_0^\pi \log(1 - z \cos \theta) \, d\theta = \log \left( \frac{1 + \sqrt{1-z^2}}{2} \right).$$
The last evaluation is elementary. It appears as entry 4.224.9 in [1]. It follows that

\[ \log \varphi_N(z) \sim \log 2 + N \log \left( \frac{z}{2} \right) - N \log \left( \frac{1 + \sqrt{1 - z^2}}{2} \right) \] (5.4)

and this is equivalent to the result.

The function

\[ A(z) = \frac{2}{1 + \sqrt{1 - 4z}} = \sum_{n=0}^{\infty} C_n z^n \] (5.5)

is the generating function for the Catalan numbers

\[ C_n = \frac{1}{n+1} \binom{2n}{n}. \] (5.6)

The final result follows directly from the expansion of Binet’s formula for the Chebyshev polynomial

\[ T_N(z) = \frac{(z - \sqrt{z^2 - 1})^N + (z + \sqrt{z^2 - 1})^N}{2}. \] (5.7)

Some standard notations are recalled. Given two sequences \( a = \{a_n\}, b = \{b_n\}, \) their convolution \( c = a * b \) is the sequence \( c = \{c_n\}, \) with

\[ c_n = \sum_{j=0}^{n} a_j b_{n-j}. \] (5.8)

The convolution power \( c^{*N} \) is the convolution of \( c \) with itself, \( N \) times.

**Theorem 5.2** For \( N \in \mathbb{N} \) fixed, the first \( N \) non-zero terms of the sequence \( q_\ell^{(N)} = 2^{\ell-1} p_\ell^{(N)} \) agree with the first \( N \) terms of the \( N \)th convolution power \( C_n^{*N} \) of the Catalan sequence:

\[ q_0^{(N)} = c_{0}^{(N)}, \; q_{N+2}^{(N)} = c_{1}^{(N)}, \; \ldots, \; q_{N+2k}^{(N)} = c_{k}^{(N)}, \; \ldots, \; q_{3N-2}^{(N)} = c_{N-1}^{(N)}. \]

In terms of generating functions, this is equivalent to

\[ \left( \sum_{n=0}^{\infty} C_n z^{2n+1} \right)^N - \sum_{\ell=0}^{\infty} q_\ell^{(N)} z^\ell \sim 2^N z^{3N}. \] (5.9)

**Acknowledgements**

The authors wish to thank referee reports on an earlier version of this paper. C.V. would like to dedicate this work to the memory of Didier Schott.

**Funding**

The second author acknowledges the partial support of NSF-DMS 1112656. The first author is a graduate student, funded in part by the same grant.
References