Asymptotic expansion for solutions of the Navier-Stokes equations with non-potential body forces

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Abstract

We study the long-time behavior of spatially periodic solutions of the Navier-Stokes equations in the three-dimensional space. The body force is assumed to possess an asymptotic expansion or, resp., finite asymptotic approximation, in either Sobolev or Gevrey spaces, as time tends to infinity, in terms of polynomial and decaying exponential functions of time. We establish an asymptotic expansion, or resp., finite asymptotic approximation, of the same type for the Leray-Hopf weak solutions. This extends the previous results, obtained in the case of potential forces, to the non-potential force case, where the body force may have different levels of regularity and asymptotic approximation. In fact, our analysis identifies precisely how the structure of the force influences the asymptotic behavior of the solutions.

1 Introduction

We study the Navier-Stokes equations (NSE) for a viscous, incompressible fluid in the three-dimensional space, $\mathbb{R}^3$. Let $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$ denote the space and time variables, respectively. Let the (kinematic) viscosity be denoted by $\nu > 0$, the velocity vector field by $u(x, t) \in \mathbb{R}^3$, the pressure by $p(x, t) \in \mathbb{R}$, and the body force by $f(x, t) \in \mathbb{R}^3$. The NSE which describe the fluid’s dynamics are given by

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u = -\nabla p + f \quad \text{on } \mathbb{R}^3 \times (0, \infty),$$

$$\text{div } u = 0 \quad \text{on } \mathbb{R}^3 \times (0, \infty).$$

(1.1)
The initial condition is
\[ u(x, 0) = u^0(x), \] (1.2)
where \( u^0(x) \) is a given divergence-free vector field.

In this paper, we focus on solutions \( u(x, t) \) and \( p(x, t) \) which are \( L \)-periodic for some \( L > 0 \). Here, a function \( g(x) \) is \( L \)-periodic if
\[ g(x + Le_j) = g(x) \quad \text{for all} \quad x \in \mathbb{R}^3, \ j = 1, 2, 3, \]
where \( \{e_1, e_2, e_3\} \) is the standard basis of \( \mathbb{R}^3 \).

By the remarkable Galilean transformation, we can assume also that \( u(x, t) \), for all \( t \geq 0 \), has zero average over the domain \( \Omega = (-L/2, L/2)^3 \). A function \( g(x) \) is said to have zero average over \( \Omega \) if
\[ \int_{\Omega} g(x) dx = 0. \] (1.3)

By rescaling the spatial and time variables, we assume throughout, without loss of generality, that \( L = 2\pi \) and \( \nu = 1 \).

**Notation.** We will use the following standard notation.

(a) In studying dynamical systems in infinite dimensional spaces, we denote, regarding (1.1) and (1.2), \( u(t) = u(\cdot, t) \), \( f(t) = f(\cdot, t) \), and \( u^0 = u^0(\cdot) \).

(b) For non-negative functions \( h(t) \) and \( g(t) \), we write
\[ h(t) = \mathcal{O}(g(t)) \text{ as } t \to \infty \quad \text{if there exist } T, C > 0 \text{ such that } h(t) \leq Cg(t), \ \forall t > T. \]

(c) The Sobolev spaces on \( \Omega \) are denoted by \( H^m(\Omega) \) for \( m = 0, 1, 2, \ldots \), each consists of functions on \( \Omega \) with distributional derivatives up to order \( m \) belonging to \( L^2(\Omega) \).

The type of asymptotic expansion that we study here is defined, in a general setting, as follows.

**Definition 1.1.** Let \( X \) be a real vector space.

(a) An \( X \)-valued polynomial is a function \( t \in \mathbb{R} \mapsto \sum_{n=1}^{d} a_n t^n \), for some \( d \geq 0 \), and \( a_n \)'s belonging to \( X \).

(b) In case \( \| \cdot \| \) is a norm on \( X \), a function \( g(t) \) from \( (0, \infty) \) to \( X \) is said to have the asymptotic expansion
\[ g(t) \sim \sum_{n=1}^{\infty} g_n(t)e^{-nt} \text{ in } X, \] (1.4)
where \( g_n(t) \)'s are \( X \)-valued polynomials, if for all \( N \geq 1 \), there exists \( \varepsilon_N > 0 \) such that
\[ \left\| g(t) - \sum_{n=1}^{N} g_n(t)e^{-nt} \right\| = \mathcal{O}(e^{-(N+\varepsilon_N)t}) \text{ as } t \to \infty. \] (1.5)
This article aims at studying the asymptotic behavior of the solution \( u(x,t) \) as \( t \to \infty \) for a certain class of forces \( f(x,t) \). In case \( f \) is a potential force, i.e., \( f(x,t) = -\nabla \phi(x,t) \), for some scalar function \( \phi \), much has been studied and is known. It is well-known that any Leray-Hopf weak solution becomes regular eventually and decays in \( H^1(\Omega) \)-norm exponentially. For more precise asymptotic behavior, Dyer and Edmunds \[2\] proved that a non-trivial, regular solution is also bounded below by an exponential function. Foias and Saut \[11\] then proved that, in bounded or periodic domains, a non-trivial, regular solution decays exponentially at an exact rate which is an eigenvalue of the Stokes operator. Furthermore, they established in \[13\] for such a solution \( u(t) \) the following asymptotic expansion, in the sense of Definition \[1.1\] in Sobolev spaces \( H^m(\Omega)^3 \), for all \( m \geq 0 \):

\[
\sum_{n=1}^{\infty} q_n(t)e^{-nt},
\]

(1.6)

where \( q_n(t) \)'s are unique polynomials in \( t \) with trigonometric polynomial values.

Recently, in \[16\], it was shown by the authors that the expansion in fact holds in Gevrey spaces. More precisely, for any \( \sigma > 0 \) and \( m \in \mathbb{N} \), there exists an \( \varepsilon_N > 0 \), for each integer \( N > 0 \), such that

\[
 \left\| e^{\sigma(-\Delta)^{1/2}} \left( u(t) - \sum_{n=1}^{N} q_n(t)e^{-nt} \right) \right\|_{H^m(\Omega)^3} = O(e^{-(N+\varepsilon_N)t}) \quad \text{as} \quad t \to \infty.
\]

(1.7)

Note that the (Gevrey) norm in estimate (1.7) is much stronger than the standard Sobolev norm in \( H^m(\Omega) \). More importantly, the simplified approach in \[16\] allows the proof to be applied to wider classes of equations; that approach will be adopted in this paper.

Regarding the case of potential forces, the interested reader is referred to \[10-14\] for deeper studies on the asymptotic expansion, its associated normalization map, and invariant nonlinear manifolds; for the associated (Poincaré-Dulac) normal form, see \[5-7\]; for its applications to statistical solutions of the NSE, decaying turbulence, and analysis of helicity, see \[3,4\]; for a result in the whole space \( \mathbb{R}^3 \), see \[17\].

The main goal of this paper is to establish (1.6) when \( f \) is not a potential function. To quickly understand the result without calling for technical details, we state it as the following meta theorem.

**Theorem 1.2 (Meta Theorem).** Assume the body force has an asymptotic expansion

\[
f(t) \sim \sum_{n=1}^{\infty} f_n(t)e^{-nt}
\]

(1.8)

in some appropriate functional spaces. Then any Leray-Hopf weak solution \( u(t) \) of (1.1) and (1.2) admits an asymptotic expansion of the form (1.6) in the same spaces.

The rigorous version of Theorem 1.2 will be detailed in Theorem 2.2 and its proof will be presented in section 4. Furthermore, when \( f(t) \) does not have such an expansion (1.8), we obtain a finite asymptotic approximation result in Theorem 2.6. Roughly speaking, if the
right-hand side of (1.8) is a finite sum, then the corresponding solution, \( u(t) \), admits a finite sum approximation in (1.6). Theorem 2.6 is also particularly suitable for the case when the force does not have arbitrarily high Sobolev regularity.

Although Theorem 1.2 is a modest extension of the Foias-Saut theory [13], it is the first of its kind to be established for NSE with non-potential force. The finite sum version, Theorem 2.6 which is a new feature only for the non-potential force case, allows asymptotic analysis of the solution even when the force has restricted regularity, and less information on its asymptotic behavior. Furthermore, our analysis shows explicitly how each term of the force’s expansion is integrated into the expansion of the solution. We also point out that the expansion is established under rather natural assumptions on the force. This technicality is not obvious because if one simply adopts the argument of Foias-Saut in [13], one would then impose conditions on \( \text{time-derivatives of all orders} \) on the force. We, nevertheless, are able to avoid these conditions and ultimately establish the claimed expansion by applying the refined method, in case of periodic domains, initiated in [16]. This, in turn, is implemented by exploiting the mechanism for the eventual high regularity of the solutions in the corresponding Sobolev and Gevrey spaces (see the asymptotic estimates in Propositions 3.2 and 3.3). We also make a slight technical improvement over [16] in obtaining the expansion in Sobolev spaces, when Gevrey regularity is not available; it requires a more elaborate bootstrapping process which is carried out in Part II of the proof of Proposition 3.3.

The results obtained in this paper are the first steps of the larger program of understanding the relation between the asymptotic expansion and the external body force. On the other hand, in spite of assuming the force to have rather simple modes of decay, further understanding of the solution’s expansion and its consequences will shed insights into the nonlinear structure of the NSE, and decaying turbulence theory. It is worth mentioning that the global well-posedness for regular solutions of NSE is still an open problem. Therefore, any such new understanding, as a rule, is welcoming. Moreover, as of yet, there have been no numerical computations made to determine the polynomials \( q_n(t) \) in (1.6). By extending the expansion to accommodate non-potential forces, our result should therefore facilitate the formulation and testing of possible numerical algorithms for their computation. Indeed, one can attempt to compute the expansion of \( \text{explicit solutions} \), particularly, those for which the nonlinear term in NSE does not vanish. These solutions are easy to generate by \( \text{specifying} \) a force, but harder to find when the projected force in the functional equation (2.2) is \( \text{given and fixed} \), albeit zero, as in the case of potential forces, see (2.4).

For the structure of this paper, Section 2 lays the necessary background for formulating the result. The main theorems are Theorems 2.2 and 2.6 while Corollary 2.5 emphasizes a scenario of finite-dimensional polynomials in (1.8) and (1.6). Section 3 obtains exponential decays for the weak solutions in Sobolev and Gevrey spaces, see Propositions 3.2 and 3.3. They are used crucially in section 4 which is devoted to the proofs of our main results.
2 Background and main results

The space $L^2(\Omega)^3$ of square (Lebesgue) integrable vector fields on $\Omega$ is a Hilbert space with the standard inner product $\langle \cdot, \cdot \rangle$ and norm $| \cdot |$ defined by

$$\langle u, v \rangle = \int_{\Omega} u(x) \cdot v(x) \, dx$$

and $|u| = \langle u, u \rangle^{1/2}$ for $u = u(\cdot)$, $v = v(\cdot)$.

We note that the notation $| \cdot |$ is also used to denote the absolute value, modulus, and, more generally, the Euclidean norm in $\mathbb{R}^n$ and $\mathbb{C}^n$, for $n \in \mathbb{N}$. Nonetheless, its meaning will be made clear by the context.

Let $V$ be the set of all $L$-periodic trigonometric polynomial vector fields which are divergence-free and have zero average over $\Omega$. Define $H$, resp. $V = \text{closure of } V$ in $L^2(\Omega)^3$, resp. $H^1(\Omega)^3$.

We use the following embeddings and identification

$$V \subset H = H' \subset V'$$

where each space is dense in the next one, and the embeddings are compact.

Let $P$ denote the orthogonal (Leray) projection in $L^2(\Omega)^3$ onto $H$. Explicitly,

$$P\left( \sum_{k \neq 0} \hat{\mathbf{u}}(k)e^{ik \cdot x} \right) = \sum_{k \neq 0} \left\{ \hat{\mathbf{u}}(k) - \left( \hat{\mathbf{u}}(k) \cdot \frac{k}{|k|} \right) \frac{k}{|k|} \right\} e^{ik \cdot x}.$$

We define the Stokes operator $A : V \to V'$ by

$$\langle Au, v \rangle_{V', V} = \langle u, v \rangle \overset{\text{def}}{=} \sum_{i=1}^{3} \langle \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \rangle,$$

for all $u, v \in V$.

As an unbounded operator on $H$, the operator $A$ has the domain $\mathcal{D}(A) = V \cap H^2(\Omega)^3$, and

$$Au = -P \Delta u = -\Delta u \in H,$$

for all $u \in \mathcal{D}(A)$.

The last identity is due to the periodic boundary conditions.

It is known that the spectrum of the Stokes operator $A$ is $\sigma(A) = \{ \lambda_j : j \in \mathbb{N} \}$, where $\lambda_j$ is strictly increasing in $j$, and is an eigenvalue of $A$. In fact, for each $j \in \mathbb{N}$, $\lambda_j = |k|^2$ for some $k \in \mathbb{Z}^3 \setminus \{0\}$. Note that since $\sigma(A) \subset \mathbb{N}$ and $\lambda_1 = 1$, the additive semigroup generated by $\sigma(A)$ is equal to $\mathbb{N}$.

For $\alpha, \sigma \in \mathbb{R}$ and $u = \sum_{k \neq 0} \hat{\mathbf{u}}(k)e^{ik \cdot x}$, define

$$A^\alpha u = \sum_{k \neq 0} |k|^{2\alpha} \hat{\mathbf{u}}(k)e^{ik \cdot x},$$

$$A^\alpha e^{\sigma A^{1/2}} u = \sum_{k \neq 0} |k|^{2\alpha} e^{\sigma |k|} \hat{\mathbf{u}}(k)e^{ik \cdot x}.$$
We then define the Gevrey spaces by

\[ G_{\alpha,\sigma} = D(A^{\alpha} e^{\sigma A^{1/2}}) \overset{\text{def}}{=} \{ u \in H : |u|_{\alpha,\sigma} \overset{\text{def}}{=} |A^{\alpha} e^{\sigma A^{1/2}} u| < \infty \}, \]

and the domain of the fractional operator \( A^{\alpha} \) by

\[ D(A^{\alpha}) = G_{\alpha,0} = \{ u \in H : |A^{\alpha} u| = |u|_{\alpha,0} < \infty \}. \]

Thanks to the zero-average condition (1.3), the norm \(|A^{m/2} u|\) is equivalent to \( \|u\|_{H^m(\Omega)^3} \) on the space \( D(A^{m/2}) \) for \( m = 0, 1, 2, \ldots \).

Note that \( D(A^{0}) = H, D(A^{1/2}) = V, \) and \( \|u\| = |\nabla u| \) is equal to \(|A^{1/2} u|\) for \( u \in V \).

Also, the spaces \( G_{\alpha,\sigma} \) are decreasing in \( \alpha \) and \( \sigma \).

Denote for \( \sigma \in \mathbb{R} \) the space \( E^{\infty,\sigma} = \bigcap_{\alpha \geq 0} G_{\alpha,\sigma} = \bigcap_{m \in \mathbb{N}} G_{m,\sigma} \).

We will say that an asymptotic expansion (1.4) holds in \( E^{\infty,\sigma} \) if it holds in \( G_{\alpha,\sigma} \) for all \( \alpha \geq 0 \).

Let us also denote by \( P_{\alpha,\sigma} \) the space of \( G_{\alpha,\sigma} \)-valued polynomials in case \( \alpha \in \mathbb{R} \), and the space of \( E^{\infty,\sigma} \)-valued polynomials in case \( \alpha = \infty \).

We define the bilinear mapping \( B : V \times V \rightarrow V' \), which is associated with the nonlinear term in the NSE, by

\[ \langle B(u, v), w \rangle_{V', V} = b(u, v, w) \overset{\text{def}}{=} \int_{\Omega} ((u \cdot \nabla)v) \cdot w \, dx, \quad \text{for all } u, v, w \in V. \]

In particular,

\[ B(u, v) = \mathcal{P}((u \cdot \nabla)v), \quad \text{for all } u, v \in D(A). \]

More precisely, for \( u = \sum_{k \neq 0} \hat{u}(k)e^{ik \cdot x} \) and \( v = \sum_{k \neq 0} \hat{v}(k)e^{ik \cdot x} \),

\[ B(u, v) = \sum_{k \neq 0} \left\{ \hat{b}(k) - \left( \hat{b}(k) \cdot \frac{k}{|k|} \right) \frac{k}{|k|} \right\} e^{ik \cdot x}, \quad \text{where } \hat{b}(k) = \sum_{m+l=k} i(\hat{u}(m) \cdot l)\hat{v}(l). \]

It is clear that

\[ B(V, V) \subset V. \quad (2.1) \]

By applying the Leray projection \( \mathcal{P} \) to (1.1) and (1.2), we rewrite the initial value problem for NSE in the functional form as

\[ \frac{du(t)}{dt} + Au(t) + B(u(t), u(t)) = \mathcal{P} f(t) \quad \text{in } V' \text{ on } (0, \infty), \quad (2.2) \]

with the initial data

\[ u(0) = u^0 \in H. \quad (2.3) \]

(See e.g. [1, 18, 20] for more details.)
Because of the projection $\mathcal{P}$ on the right-hand side of (2.2), it is convenient to assume, without loss of generality that the force belongs to $H$. Then, we have

$$\mathcal{P}f(t) = f(t) \text{ in (2.2)}.$$  

In the case $f(x, t)$ is a potential force, then, by the Helmholtz-Leray decomposition,

$$\mathcal{P}f(t) \equiv 0 \text{ in the functional equation (2.2).}$$ (2.4)

In dealing with weak solutions of (2.2), we follow the presentation in [8] and use the results there.

**Definition 2.1.** Let $f \in L^2_{\text{loc}}([0, \infty), H)$. A Leray-Hopf weak solution $u(t)$ of (2.2) is a mapping from $[0, \infty)$ to $H$ such that

$$u \in C([0, \infty), H_w) \cap L^2_{\text{loc}}([0, \infty), V), \quad u' \in L^{4/3}_{\text{loc}}([0, \infty), V'),$$ (2.5)

and satisfies

$$\frac{d}{dt} \langle u(t), v \rangle + \langle u(t), v \rangle + b(u(t), u(t), v) = \langle f(t), v \rangle$$ (2.6)

in the distribution sense in $(0, \infty)$, for all $v \in V$, and the energy inequality

$$\frac{1}{2} |u(t)|^2 + \int_{t_0}^t \|u(\tau)\|^2 d\tau \leq \frac{1}{2} |u(t_0)|^2 + \int_{t_0}^t \langle f(\tau), u(\tau) \rangle d\tau$$ (2.7)

holds for $t_0 = 0$ and almost all $t_0 \in (0, \infty)$, and all $t \geq t_0$.

We will say that a Leray-Hopf weak solution $u(t)$ is regular if $u \in C([0, \infty), V)$.

Above, $H_w$ is the topological vector space $H$ with the weak topology.

This definition of the Leray-Hopf weak solutions with the choice of the energy inequality (2.7) is, in fact, equivalent to the weak solutions used in [8, Chapter II, section 7], see e.g. Remark 1(e) of [9] for the explanations.

**Basic Assumption.** It is assumed throughout the paper that the function $f(t)$ belongs to $L^\infty_{\text{loc}}([0, \infty), H)$.

This assumption guarantees the existence of the Leray-Hopf weak solutions for any $u^0 \in H$, see e.g. [8]. It is not strict since we later consider decaying $f(t)$ in even Sobolev norms.

Regarding the problem of finding asymptotic expansions for solutions of the NSE, it is natural, at this stage, to study the class of functions $f(t)$ which have similar asymptotic behavior as $u(t)$ in (1.6). We specify the condition on $f(t)$ more precisely in the next theorem, which is our first main result.

**Theorem 2.2** (Asymptotic expansion). Assume that there exist a number $\sigma_0 \geq 0$ and polynomials $f_n \in \mathcal{P}^\infty, \sigma_0$, for all $n \geq 1$, such that $f(t)$ has the asymptotic expansion

$$f(t) \sim \sum_{n=1}^\infty f_n(t)e^{-nt} \text{ in } E^\infty, \sigma_0.$$ (2.8)
Let \( u(t) \) be a Leray-Hopf weak solution of (2.2) and (2.3). Then there exist polynomials \( q_n \in P_{\infty,\sigma_0}^\infty, \) for all \( n \geq 1, \) such that \( u(t) \) has the asymptotic expansion
\[
u(t) \sim \sum_{n=1}^{\infty} q_n(t)e^{-nt} \quad \text{in } E_{\infty,\sigma_0}^\infty.
\] (2.9)

Moreover, the mappings
\[
u_n(t) \overset{\text{def}}{=} q_n(t)e^{-nt} \quad \text{and} \quad F_n(t) \overset{\text{def}}{=} f_n(t)e^{-nt},
\] (2.10)
satisfy the following ordinary differential equations in the space \( E_{\infty,\sigma_0}^\infty \)
\[rac{d}{dt}u_n(t) + Au_n(t) + \sum_{k+m=n} B(u_k(t), u_m(t)) = F_n(t), \quad t \in \mathbb{R},
\] (2.11)
for all \( n \geq 1. \)

Regarding equation (2.11), when \( n = 1, \) the sum on its left-hand side is empty, hence the equation reads as
\[rac{d}{dt}u_1(t) + Au_1(t) = F_1(t).
\] (2.12)

Remark 2.3. Observe that since the expansion (2.8) is an infinite sum, it immediately implies the following remainder estimate:
\[
\left| f(t) - \sum_{n=1}^{N} f_n(t)e^{-nt} \right|_{\alpha,\sigma_0} \leq \left| f_{N+1}(t)e^{-(N+1)t} \right|_{\alpha,\sigma_0} + \left| f(t) - \sum_{n=1}^{N+1} f_n(t)e^{-nt} \right|_{\alpha,\sigma_0}
\]
\[
= O(e^{-(N+\varepsilon)t}) + O(e^{-(N+1+\delta_{N+1},\alpha)t}),
\]
which holds for all \( N \geq 1, \alpha \geq 0, \varepsilon \in (0,1) \) and some \( \delta_{N+1,\alpha} \in (0,1). \) Therefore, we have for all \( N \geq 1, \alpha \geq 0 \) that
\[
\left| f(t) - \sum_{n=1}^{N} f_n(t)e^{-nt} \right|_{\alpha,\sigma_0} = O(e^{-(N+\varepsilon)t}) \quad \text{as } t \to \infty, \quad \forall \varepsilon \in (0,1).
\] (2.13)

Similarly, the expansion (2.9) implies for any \( N \geq 1 \) and \( \alpha \geq 0 \) that
\[
\left| u(t) - \sum_{n=1}^{N} q_n(t)e^{-nt} \right|_{\alpha,\sigma_0} = O(e^{-(N+\varepsilon)t}) \quad \text{as } t \to \infty, \quad \forall \varepsilon \in (0,1).
\] (2.14)
(In fact, the same argument applies to the general expansion (1.4) so that \( \varepsilon_N \) in (1.5) can be taken arbitrarily in \( (0,1). \))

Remark 2.4. The following additional remarks on Theorem 2.2 are in order.
(a) We do not require the time derivative $d^m f/dt^m$, for all $m \in \mathbb{N}$, to have any kind of expansion. Indeed, this rather stringent requirement would be imposed if one adapts Foias-Saut’s original proof. It is due to higher regularity of the solutions for large time, in the particular case of periodic domains, either in Sobolev or Gevrey spaces. In the case of Sobolev spaces ($\sigma_0 = 0$), it requires a bootstrapping scheme to gradually increase the regularity to any needed level. In the case of the Gevrey spaces ($\sigma_0 > 0$), the effect is immediate, hence the proof is shorter. (For the related Gevrey norm techniques, see \cite{15,16,19} and references therein.)

(b) The equations (2.11) determine the polynomials $q_n(t)$’s and indicate the interactions on all scales between the body force and the nonlinear terms in NSE. Even though the solution $u(t)$ decays to zero, such interactions are complicated.

(c) Condition (2.8) is easily satisfied for any finite sum

$$f(t) = \sum_{n=1}^{N} f_n(t)e^{-nt},$$

for some fixed $N \geq 1$, and polynomials $f_n(t)$’s belonging to $\mathcal{V}$. Even in this case, the result in Theorem 2.2 is new and the expansion (2.9) for $u(t)$ can still be an infinite sum.

(d) If the expansion (2.8) for $f(t)$ holds in $G_{0,\sigma}$ for some $\sigma > 0$, then it holds in $E^{\infty,\sigma}$ for any $\sigma_0 \in (0, \sigma_1)$, and hence, by Theorem 2.2 the solution $u(t)$ admits the expansion (2.9) in $E^{\infty,\sigma}$ for any $\sigma_0 \in (0, \sigma_1)$.

(e) The equations (2.11) in fact are linear systems of ordinary differential equations (ODEs) in infinite-dimensional spaces. They form an integrable system in the sense that it can be recursively solved by the variation of constants formula. Moreover, each solution $u_n(t)$ of the form (2.10) is uniquely determined provided that $R_n u_n(0) = \xi_n \in R_n H$ is given.

In the following simple scenario, the ODEs (2.11) are finite-dimensional systems, which make them more accessible for deeper study, and, in particular, for numerical computations.

Corollary 2.5. If all $f_n(t)$’s in Theorem 3.2 are $\mathcal{V}$-valued polynomials, then so are the polynomials $q_n(t)$’s in the expansion (2.9), and consequently, the equations (2.11) are systems of linear ODEs in finite-dimensional spaces.

Next is the paper’s second main result which deals with the case when $f(t)$ does not possess an asymptotic expansion (2.8), but rather a finite asymptotic approximation. Then it is proved that the weak solution also admits a finite asymptotic approximation of the same type.

Theorem 2.6 (Finite asymptotic approximation). Suppose there exist an integer $N_* \geq 1$, real numbers $\sigma_0 \geq 0$, $\mu_* \geq \alpha_* \geq N_*/2$, and, for any $1 \leq n \leq N_*$, numbers $\delta_n \in (0, 1)$ and polynomials $f_n \in \mathcal{P}^{\mu_*,\sigma_0}$, such that

$$\left| f(t) - \sum_{n=1}^{N} f_n(t)e^{-nt} \right|_{\alpha_*,\sigma_0} = O(e^{-(N+\delta_N)t}) \quad \text{as} \quad t \to \infty,$$

(2.15)
for $1 \leq N \leq N_*$, where

$$\mu_n = \mu_\ast - (n - 1)/2, \quad \alpha_n = \alpha_\ast - (n - 1)/2.$$ 

Let $u(t)$ be a Leray-Hopf weak solution of (2.2) and (2.3).

(i) Then there exist polynomials $q_n \in P^{\mu_n,1,\sigma_0}$, for $1 \leq n \leq N_*$, such that one has for $1 \leq N \leq N_*$ that

$$\left| u(t) - \sum_{n=1}^{N} q_n(t)e^{-nt} \right|_{\alpha_N,\sigma_0} = \mathcal{O}(e^{-(N+\varepsilon)t}) \quad \text{as} \ t \to \infty, \quad \forall \varepsilon \in (0, \delta^*_N), \quad (2.16)$$

where $\delta^*_N = \min\{\delta_1, \delta_2, \ldots, \delta_N\}$.

Moreover, the ODEs (2.11) hold in the corresponding space $G^{\mu_n,\sigma_0}$ for $1 \leq n \leq N_*$.

(ii) In particular, if all $f_n(t)$’s belong to $\mathcal{V}$, resp., $E^{\infty,\sigma_0}$, then so do all $q_n(t)$’s, and the ODEs (2.11) hold in $\mathcal{V}$, resp., $E^{\infty,\sigma_0}$.

Regarding the finite approximation (2.15), in addition to its sum having only finitely many terms ($N \leq N_*$), the force $f(t)$ now has much less regularity compared to the expansion (2.8). This, therefore, determines the regularity of the solution $u(t)$ and its asymptotic approximation (2.16). Such dependence was worked out in detail in the above theorem.

It is worth noticing that Theorem 2.6 is stronger than Theorem 2.2. Nevertheless, the proof of Theorem 2.6, in the current presentation, is adapted from that of Theorem 2.2.

3 Exponential decay in Gevrey and Sobolev spaces

This section prepares for the proofs in section 4. Particularly, we will derive the exponential decay for weak solutions in both Gevrey and Sobolev spaces.

First, we have a few basic inequalities concerning the Sobolev and Gevrey norms. It is elementary to see that

$$\max_{x \geq 0}(x^{2\alpha}e^{-\sigma x}) = \left(\frac{2\alpha}{\sigma}\right)^{2\alpha} \quad \text{for any} \ \sigma, \alpha > 0.$$ 

Applying this inequality, one easily obtains

$$|A^\alpha u| = |(A^\alpha e^{-\sigma A^{1/2}})e^{\sigma A^{1/2}} u| \leq \left(\frac{2\alpha}{\sigma}\right)^{2\alpha} |e^{\sigma A^{1/2}} u| \quad \forall \alpha, \sigma > 0. \quad (3.1)$$

We recall a key estimate for the Gevrey norm of the bilinear form (cf. Lemma 2.1 in [16]), which is based mainly on the work of Foias and Temam [15].

**Lemma 3.1.** Let $\sigma \geq 0$ and $\alpha \geq 1/2$. There exists an absolute constant $K > 1$, independent of $\alpha, \sigma$, such that

$$|B(u, v)|_{\alpha,\sigma} \leq K^\alpha |u|_{\alpha+1/2,\sigma}|v|_{\alpha+1/2,\sigma} \quad \forall u, v \in G_{\alpha+1/2,\sigma}. \quad (3.2)$$
Although inequality (3.2) is not sharp, it is very convenient for our calculations below and will be sufficient for our purposes.

As a consequence of Lemma 3.1, we have

\[ B(G_{\alpha+1/2,\sigma}, G_{\alpha+1/2,\sigma}) \subset G_{\alpha,\sigma} \quad \text{for } \alpha \geq 1/2, \sigma \geq 0, \]  

(3.3)

\[ B(E^{\infty,\sigma}, E^{\infty,\sigma}) \subset E^{\infty,\sigma} \quad \text{for } \sigma \geq 0. \]  

(3.4)

Next, we prove a small data result, which establishes the global existence of the solution in Gevrey spaces and its exponential decay as time goes to infinity.

**Proposition 3.2.** Let \( \delta \in (0, 1), \lambda \in (1 - \delta, 1] \) and \( \sigma \geq 0, \alpha \geq 1/2 \). Define the positive numbers \( C_0 = C_0(\alpha, \delta) \) and \( C_1 = C_1(\alpha, \delta, \lambda) \) by

\[ C_0(\alpha, \delta) = \begin{cases} \frac{\delta}{6K^\alpha}, & \text{if } \sigma > 0, \\ \frac{\delta}{4K^\alpha}, & \text{if } \sigma = 0 \end{cases}, \quad C_1(\alpha, \delta, \lambda) = \begin{cases} \frac{2}{\sqrt{3}} \sqrt{\delta(\lambda - 1 + \delta)}, & \text{if } \sigma > 0, \\ \sqrt{\delta(\lambda - 1 + \delta)}, & \text{if } \sigma = 0. \end{cases} \]

Suppose

\[ |A^\alpha u^0| \leq C_0, \]  

(3.5)

and

\[ |f(t)|_{\alpha-1/2,\sigma} \leq C_1 e^{-\lambda t}, \quad \forall t \geq 0. \]  

(3.6)

Then there exists a unique solution \( u(t) \) of (2.2) and (2.3) that satisfies

\[ u \in C([0, \infty), \mathcal{D}(A^\alpha)) \]

and

\[ |u(t)|_{\alpha,\sigma} \leq \sqrt{2}C_0 e^{-(1-\delta)t}, \quad \forall t \geq t_*, \]  

(3.7)

where \( t_* = 6\sigma/\delta \). Moreover, one has for all \( t \geq t_* \) that

\[ \int_t^{t+1} |A^{\alpha+1/2} u(\tau)|^2 d\tau \leq \frac{2C_0^2}{1-\delta} e^{-2(1-\delta)t}. \]  

(3.8)

**Proof.** While the estimates below are formal, they can be justified by performing them at the level of the Galerkin approximation and then passing to the limit. The estimates will hold for the unique, regular solution \( u(t) \).

**Part I: case \( \sigma > 0.** Let \( \varphi(t) \) be a function in \( C^\infty(\mathbb{R}) \) such that

\[ \varphi((-\infty, 0]) = \{0\}, \quad \varphi([0, t_*]) = [0, \sigma], \quad \varphi([t_*, \infty)) = \{\sigma\}, \]

and

\[ 0 < \varphi'(t) < 2\sigma/t_* = \delta/3 \quad \text{for all } t \in (0, t_*). \]

From equation (2.22), we have

\[ \frac{d}{dt} (A^\alpha e^{\varphi(t) A^{1/2}} u(t)) = A^\alpha e^{\varphi(t) A^{1/2}} (-Au - B(u, u) + f) + \varphi'(t) A^{1/2} A^\alpha e^{\varphi(t) A^{1/2}} u. \]  

(3.9)
Taking inner product of the equation (3.9) with $A^\alpha e^{\varphi(t)}A^{1/2}u(t)$ gives

$$
\frac{1}{2} \frac{d}{dt} |u|_{\alpha, \varphi(t)}^2 + |A^{1/2}u|_{\alpha, \varphi(t)}^2 = \varphi'(t) \langle A^{2\alpha+1/2}e^{2\varphi(t)}A^{1/2}u, u \rangle
\quad - \langle A^\alpha e^{\varphi(t)}A^{1/2}B(u, u), A^\alpha e^{\varphi(t)}A^{1/2}u \rangle + \langle A^{\alpha-1/2}e^{\varphi(t)}A^{1/2}f, A^{\alpha+1/2}e^{\varphi(t)}A^{1/2}u \rangle.
$$

Applying the Cauchy-Schwarz inequality, then Lemma 3.1 to the second term on the right-hand side, we obtain

$$
\frac{1}{2} \frac{d}{dt} |u|_{\alpha, \varphi(t)}^2 + |A^{1/2}u|_{\alpha, \varphi(t)}^2 \leq \varphi'(t)|u|_{\alpha+1/2, \varphi(t)}^2 + K^\alpha |A^{1/2}u|_{\alpha, \varphi(t)}^2|u|_{\alpha, \varphi(t)} + |f(t)|_{\alpha-1/2, \varphi(t)}|u|_{\alpha+1/2, \varphi(t)},
\quad \leq \frac{\delta}{3} |u|_{\alpha+1/2, \varphi(t)}^2 + K^\alpha |A^{1/2}u|_{\alpha, \varphi(t)}^2|u|_{\alpha, \varphi(t)} + \frac{3}{4\delta} |f(t)|_{\alpha-1/2, \varphi(t)}^2 + \frac{\delta}{3} |u|_{\alpha+1/2, \varphi(t)}^2.
$$

This implies

$$
\frac{1}{2} \frac{d}{dt} |u|_{\alpha, \varphi(t)}^2 + \left(1 - \frac{2\delta}{3} - K^\alpha|u|_{\alpha, \varphi(t)} \right) |A^{1/2}u|_{\alpha, \varphi(t)}^2 \leq \frac{3}{4\delta} |f(t)|_{\alpha-1/2, \varphi(t)}^2. \tag{3.10}
$$

Let $T \in (0, \infty)$. Note that $|u(0)|_{\alpha, \varphi(0)} = |A^\alpha u^0| < 2C_0$. Assume that

$$
|u(t)|_{\alpha, \varphi(t)} \leq 2C_0, \quad \forall t \in [0, T). \tag{3.11}
$$

Then for $t \in (0, T)$, we have from (3.10) and (3.6) that

$$
\frac{d}{dt} |u|_{\alpha, \varphi(t)}^2 + 2(1 - \delta)|A^{1/2}u|_{\alpha, \varphi(t)}^2 \leq \frac{3}{2\delta} |f(t)|_{\alpha-1/2, \varphi(t)}^2 \leq \frac{3C_1^2}{2\delta} e^{-2\lambda t}. \tag{3.12}
$$

Applying Gronwall’s inequality in (3.12) yields for all $t \in (0, T)$ that

$$
|u(t)|_{\alpha, \varphi(t)}^2 \leq e^{-2(1-\delta)t}|u|_{\alpha, \varphi(t)}^2 + \frac{3C_1^2}{2\delta} e^{-2(1-\delta)t} \int_0^t e^{2(1-\delta)\tau} e^{-2\lambda \tau} d\tau
\leq e^{-2(1-\delta)t}|u|_{\alpha, \varphi(t)}^2 + \frac{3C_1^2}{4\delta(\lambda-1+\delta)} e^{-2(1-\delta)t}
= \left(|u|_{\alpha, \varphi(t)}^2 + C_0^2\right) e^{-2(1-\delta)t}.
$$

Combining this with condition (3.5) for the initial data, we obtain

$$
|u(t)|_{\alpha, \varphi(t)}^2 \leq 2C_0^2 e^{-2(1-\delta)t},
$$

which gives

$$
|u(t)|_{\alpha, \varphi(t)} \leq \sqrt{2}C_0 e^{-1(1-\delta)t}. \tag{3.13}
$$

In particular, letting $t \to T^-$ in (3.13) yields

$$
\lim_{t \to T^-} |u(t)|_{\alpha, \varphi(t)} \leq \sqrt{2}C_0 < 2C_0.
$$
By the standard contradiction argument, we have that the inequality in (3.11) holds for all \( t > 0 \), and consequently, so does (3.13). Since \( \varphi(t) = \sigma \) for \( t \geq t_* \), the desired estimate (3.7) follows (3.13).

For \( t \geq t_* \), integrating (3.12) from \( t \) to \( t + 1 \) gives

\[
2(1 - \delta) \int_t^{t+1} |A^{\alpha+1/2} u(\tau)|^2 d\tau \leq |A^{\alpha} u(t)|^2 + \frac{3C^2_1}{\delta} \int_t^{t+1} e^{-2\lambda \tau} d\tau
\]

\[
\leq 2C^2_0 e^{-2(1-\delta)t} + \frac{3C^2_1}{2\delta \lambda} e^{-2\lambda t} \leq 2C^2_0 e^{-2(1-\delta)t} \left( 1 + \frac{\lambda - 1 + \delta}{\lambda} \right)
\]

Then estimate (3.8) follows.

**Part II: case \( \sigma = 0 \)**. The proof is similar to Part I without using the function \( \varphi(t) \). Here, we perform necessary calculations. First, using Sobolev norms, we have

\[
\frac{1}{2} \frac{d}{dt} |A^{\alpha} u|^2 + \left( 1 - \frac{\delta}{2} - K\alpha |A^{\alpha} u| \right) |A^{\alpha+1/2} u|^2 \leq \frac{1}{2\delta} |A^{\alpha-1/2} f|^2.
\]

(3.14)

As long as \( |A^{\alpha} u(t)| \leq 2C_0 = \delta/(2K\alpha) \) in \([0, T)\) for some \( T \in (0, \infty) \), we have for \( t \in (0, T) \) that

\[
|A^{\alpha} u(t)|^2 \leq |A^{\alpha} u(0)|^2 e^{-2(1-\delta)t} + \frac{C^2_1 e^{-2(1-\delta)t}}{\delta} \int_0^t e^{-2(\lambda - \delta + 1)\tau} d\tau
\]

\[
\leq \left( C^2_0 + \frac{C^2_1}{\delta (\lambda - \delta + 1)} \right) e^{-2(1-\delta)t} = 2C^2_0 e^{-2(1-\delta)t}.
\]

This implies \( T = \infty \) and then also proves (3.7). Now, using (3.7) for the second term on the left-hand side of (3.14), and then integrating in time gives

\[
2(1 - \delta) \int_t^{t+1} |A^{\alpha+1/2} u|^2 d\tau \leq |A^{\alpha} u(t)|^2 + \frac{1}{\delta} \int_t^{t+1} |A^{\alpha-1/2} f|^2 d\tau
\]

\[
\leq 2C^2_0 e^{-2(1-\delta)t} + \frac{C^2_1 e^{-2\lambda t}}{2\delta \lambda} \leq \frac{5}{2} C^2_0 e^{-2(1-\delta)t}.
\]

This implies (3.8), and the proof is complete.

Considering decaying forces, we assume at the moment up to Proposition 3.3 that there are numbers \( M_*, \kappa_0 > 0 \) such that

\[
|f(t)| \leq M_* e^{-(1+\kappa_0)t/2}, \quad \forall t \geq 0.
\]

(3.15)

We recall estimate (A.39) of Chapter II for Leray-Hopf weak solutions (under the Basic Assumption),

\[
|u(t)|^2 \leq e^{-t} |u_0|^2 + e^{-t} \int_0^t e^\tau |f(\tau)|^2 d\tau, \quad \forall t > 0.
\]

It then follows from (3.15) that

\[
|u(t)|^2 \leq e^{-t} (|u_0|^2 + M^2_*/\kappa_0), \quad \forall t > 0.
\]

(3.16)
By applying the Cauchy-Schwarz, Cauchy, and Poincaré inequalities to the last term on the right-hand side of (2.7), upon simplifying we obtain

\[ |u(t)|^2 + \int_{t_0}^{t} |u(\tau)|^2 d\tau \leq |u(t_0)|^2 + \int_{t_0}^{t} |f(\tau)|^2 d\tau, \]

for \( t_0 = 0 \) and almost all \( t_0 \in (0, \infty) \), and all \( t \geq t_0 \). Using (3.16) for \( |u(t_0)|^2 \), and, again, (3.15) yields

\[ \int_{t_0}^{t+1} |u(\tau)|^2 d\tau \leq e^{-t_0} \left( |u_0|^2 + \frac{M^2}{\kappa_0} \right) + \frac{1}{1 + \kappa_0} M^2 e^{-(1 + \kappa_0)t_0} \leq e^{-t_0} \left( |u_0|^2 + \frac{2M^2}{\kappa_0} \right). \]  

(3.17)

For any \( t \geq 0 \), let \( \{t_n\}_{n=1}^{\infty} \) be a sequence in \((0, \infty)\) converging to \( t \) such that (3.17) holds for \( t_0 = t_n \). Then letting \( n \to \infty \) gives

\[ \int_{t}^{t+1} |u(\tau)|^2 d\tau \leq e^{-t} \left( |u_0|^2 + \frac{2M^2}{\kappa_0} \right). \]  

(3.18)

**Proposition 3.3.** Assume (3.15) and, additionally, that there are \( \sigma \geq 0 \), \( \alpha \geq 1/2 \) and \( \lambda_0 \in (0, 1) \) such that

\[ |f(t)|_{\alpha, \sigma} = O(e^{-\lambda_0 t}) \quad \text{as} \quad t \to \infty. \]  

(3.19)

Let \( u(t) \) be a Leray-Hopf weak solution of (2.2). Then for any \( \delta \in (1 - \lambda_0, 1) \), there exists \( T_* > 0 \) such that \( u(t) \) is a regular solution of (2.2) on \([T_*, \infty)\), and one has for all \( t \geq 0 \) that

\[ |u(T_* + \tau)|_{\alpha + 1/2, \sigma} \leq K^{-\alpha - 1} e^{-(1 - \delta^*) \tau}, \]  

(3.20)

\[ |B(u(T_* + \tau), u(T_* + \tau))|_{\alpha, \sigma} \leq K^{-\alpha - 1} e^{-2(1 - \delta^*) \tau}, \]  

(3.21)

where \( K \) is the constant in Lemma 3.1.

**Proof.** First, we note that (3.21) is a direct consequence of (3.20). Indeed, applying Lemma 3.1 with the use of (3.20), we have for \( t \geq 0 \),

\[ |B(u(T_* + \tau), u(T_* + \tau))|_{\alpha, \sigma} \leq K^\alpha |u(T_* + \tau)|_{\alpha + 1/2, \sigma}^2 \leq K^\alpha \left( \frac{e^{-(1 - \delta^*) \tau}}{K^{\alpha + 1/2}} \right)^2, \]

which yields (3.21).

We focus on proving (3.20) now. Take \( \kappa \in (0, \kappa_0) \) such that

\[ \lambda \overset{\text{def}}{=} \frac{1 + \kappa}{2} \in \left( \frac{1}{2}, \frac{1 + \kappa_0}{2} \right), \]  

(3.22)

and define

\[ \lambda' = \frac{1 - \delta + \lambda_0}{2} \in (1 - \delta, \lambda_0). \]  

(3.23)

We consider each case \( \sigma > 0 \) and \( \sigma = 0 \) separately.

**(i) Case \( \sigma > 0 \).**

**Step 1.** By (3.18) and (3.15), there exists \( t_0 > 0 \) such that

\[ |A^{1/2} u(t_0)| < C_0(1/2, 1/2), \]
\[ |f(t_0 + t)|_{0, \sigma} \leq C_1(1/2, 1/2, \lambda) e^{-\lambda t}, \quad \forall t \geq 0. \]

Applying Proposition 3.2 to \( u(t_0 + \cdot) \), \( f(t_0 + \cdot) \), \( \alpha = 1/2 \), \( \delta = 1/2 \), and \( \lambda \) defined in (3.22) results in

\[ |u(t_0 + t)|_{1/2, \sigma} \leq \sqrt{2} C_0(1/2, 1/2) e^{-t/2} \leq K^{-1/2} e^{-t/2}, \quad \forall t \geq t_* \overset{\text{def}}{=} 12\sigma. \]

Then by (3.4), we have for all \( t \geq t_* \) that

\[ |A^{\alpha+1/2} u(t_0 + t)| \leq \left( \frac{2\alpha + 1}{e\sigma} \right)^{2\alpha+1} |e^{2t} u(t_0 + t)| \leq \left( \frac{2\alpha + 1}{e\sigma} \right)^{2\alpha+1} |u(t_0 + t)|_{1/2, \sigma} \leq \left( \frac{2\alpha + 1}{e\sigma} \right)^{2\alpha+1} K^{-1/2} e^{-t/2}. \] (3.24)

**Step 2.** From (3.24) and (3.19) we deduce that there is a sufficiently large \( T > t_0 + t_* \) so that

\[ |A^{\alpha+1/2} u(T)| \leq C_0(\alpha + 1/2, \delta), \]

\[ |f(T + t)|_{\alpha, \sigma} \leq C_1(\alpha + 1/2, \delta, \lambda') e^{-\lambda't} \quad \forall t \geq 0. \]

Applying Proposition 3.2 again to \( u(T + \cdot) \), \( \alpha := \alpha + 1/2 \), and \( \lambda := \lambda' \) defined by (3.23), we obtain that there is \( T_* > T + t_* \) such that

\[ |u(T_* + t)|_{\alpha+1/2, \sigma} \leq \sqrt{2} C_0(\alpha + 1/2, \delta) e^{-(1-\delta)t} \leq \frac{1}{K^{\alpha+1/2}} e^{-(1-\delta)t} \quad \forall t \geq 0. \]

This yields (3.20) and completes the proof of Case (i).

(ii) **Case** \( \sigma = 0 \). We will apply Proposition 3.2 recursively to gain the exponential decay for \( u(t) \) in higher Sobolev norms.

For \( j \in \mathbb{N} \), suppose

\[ \lim_{t \to \infty} \int_t^{t+1} |A^j u(\tau)|^2 d\tau = 0, \] (3.25)

and

\[ |A^{j+1} f(t)| = O(e^{-\lambda_0 t}) \quad \text{as} \quad t \to \infty. \] (3.26)

Then there is \( T > 0 \) so that

\[ |A^{j/2} u(T)| \leq C_0(j/2, \delta), \]

\[ |A^{j/2-1/2} f(T + t)| \leq C_1(j/2, \delta, \lambda') e^{-\lambda't} \quad \forall t \geq 0. \]

Applying Proposition 3.2 to \( u(T + \cdot) \), \( \alpha := j/2, \sigma := 0 \), and \( \lambda := \lambda' \), we obtain

\[ |A^{j/2} u(T + t)| \leq \sqrt{2} C_0(j/2, \delta) e^{-(1-\delta)t} \leq \frac{1}{K^{j/2}} e^{-(1-\delta)t} \quad \forall t \geq 0, \]

and

\[ \lim_{t \to \infty} \int_t^{t+1} |A^{(j+1)/2} u(\tau)|^2 d\tau = 0. \] (3.27)

Note, by (3.18), that (3.25) holds true for \( j = 1 \). Repeat the arguments from (3.25) to (3.27) for \( j = 1, 2, \ldots, m \), where \( m \) is a number in \( \mathbb{N} \cup \{0\} \) such that

\[ \alpha \leq m/2 < \alpha + 1/2. \] (3.28)
Note that $\alpha \geq 1/2$ implies $m \geq 1$. Also, we have from (3.28) that $(m-1)/2 < \alpha$, hence, by (5.11), condition (3.26) is satisfied for $j = 1, 2, \ldots, m$. Particularly, when $j = m$ we obtain from (3.27) that

$$\int_{t}^{t+1} |A^{(m+1)/2}u(\tau)|^2d\tau = O(e^{-t}) \text{ as } t \to \infty.$$  

Since $\alpha \leq m/2$, this yields

$$\int_{t}^{t+1} |A^{\alpha+1/2}u(\tau)|^2d\tau = O(e^{-t}) \text{ as } t \to \infty.$$  

(3.29)

Using (3.29) in place of (3.24), we can proceed as in Step 2 of part (i) and obtain (3.20). The proof is complete.

4 Proofs of main results

We will use the following elementary identities: for $\beta > 0$, integer $d \geq 0$, and any $t \in \mathbb{R}$,

$$\int_{-\infty}^{t} \tau^d e^{\beta \tau} d\tau = \frac{e^{\beta t}}{\beta} \sum_{n=0}^{d} \frac{(-1)^{d-n}d!}{n!\beta^{d-n}} t^n,$$  

(4.1)

$$\int_{t}^{\infty} \tau^d e^{-\beta \tau} d\tau = \frac{e^{-\beta t}}{\beta} \sum_{n=0}^{d} \frac{d!}{n!\beta^{d-n}} t^n.$$  

(4.2)

The next lemma is a building block of the construction of the polynomials $q_n(t)$’s. It summarizes and reformulates the facts used in [13] and [14, Lemma 3.2], see also [16].

Definition 4.1. Let $X$ be a Banach space with its dual $X'$. Let $u(t)$ and $g(t)$ be functions in $L_{\text{loc}}^1([0, \infty), X)$. We say $g(t)$ is the $X$-valued distribution derivative of $u(t)$, and denote $g = u'$, if

$$\frac{d}{dt} \langle u(t), v \rangle = \langle g(t), v \rangle \text{ in the distribution sense on } (0, \infty), \forall v \in X',$$  

(4.3)

where $\langle \cdot, \cdot \rangle$ in (4.3) denotes the usual duality pairing between an element of $X$ and $X'$.

Lemma 4.2. Let $(X, \| \cdot \|)$ be a Banach space. Suppose $y(t)$ is a function in $C([0, \infty), X)$ that solves the following ODE

$$y'(t) + \beta y(t) = p(t) + g(t)$$  

in the $X$-valued distribution sense on $(0, \infty)$. Here, $\beta \in \mathbb{R}$ is a fixed constant, $p(t)$ is an $X$-valued polynomial in $t$, and $g \in L_{\text{loc}}^1([0, \infty), X)$ satisfies

$$\|g(t)\| \leq Me^{-\delta t} \forall t \geq 0, \text{ for some } M, \delta > 0.$$
Define $q(t)$ for $t \in \mathbb{R}$ by

$$q(t) = \begin{cases} 
  e^{-\beta t} \int_{-\infty}^{t} e^{\beta \tau} p(\tau) d\tau & \text{if } \beta > 0, \\
  y(0) + \int_{0}^{\infty} g(\tau) d\tau + \int_{0}^{t} p(\tau) d\tau & \text{if } \beta = 0, \\
  -e^{-\beta t} \int_{t}^{\infty} e^{\beta \tau} p(\tau) d\tau & \text{if } \beta < 0.
\end{cases} \quad (4.4)$$

Then $q(t)$ is an $X$-valued polynomial of degree at most $\deg(p) + 1$ that satisfies

$$q'(t) + \beta q(t) = p(t), \quad t \in \mathbb{R}, \quad (4.5)$$

and the following estimates hold:

(i) If $\beta > \delta$ then

$$\|y(t) - q(t)\| \leq \left(\|y(0) - q(0)\| + \frac{M}{\beta - \delta}\right) e^{-\delta t}, \quad t \geq 0. \quad (4.6)$$

(ii) If $\beta = 0$ then

$$\|y(t) - q(t)\| \leq \frac{M}{\delta} e^{-\delta t}, \quad t \geq 0. \quad (4.7)$$

(iii) If $\beta < 0$ and

$$\lim_{t \to \infty} (e^{\beta t} y(t)) = 0, \quad (4.8)$$

then

$$\|y(t) - q(t)\| \leq \frac{M}{|\beta| + \delta} e^{-\delta t}, \quad t \geq 0. \quad (4.9)$$

Proof. The fact that $q(t)$ is a polynomial in $t$ follows the identities (4.1) and (4.2). The equation (4.5) obviously results from the definition (4.4) of $q(t)$. It remains to prove estimates (4.6), (4.7) and (4.9).

Let $z(t) = y(t) - q(t)$, then

$$z'(t) + \beta z(t) = g(t)$$

in the $X$-valued distribution sense on $(0, \infty)$.

Multiplying this equation by $e^{\beta t}$ yields

$$(e^{\beta t} z(t))' = e^{\beta t} g(t)$$

in the $X$-valued distribution sense on $(0, \infty). \quad (4.10)$$

For $t_0 \geq 0$, it follows (4.11) and [20] Ch. III, Lemma 1.1] that

$$e^{\beta t} z(t) = \xi + \int_{t_0}^{t} e^{\beta \tau} g(\tau) d\tau, \quad (4.11)$$

for some $\xi \in X$ and almost all $t \in (t_0, \infty)$.

Since $e^{\beta t} z(t)$ is continuous on $[0, \infty)$ and $e^{\beta t} g(t) \in L^1_{\text{loc}}([0, \infty))$, we have $\xi = e^{\beta t_0} z(t_0)$ and equation (4.11) holds for all $t \geq t_0$. Hence, we obtain the standard variation of constant formula

$$z(t) = e^{-\beta(t-t_0)} z(t_0) + e^{-\beta t} \int_{t_0}^{t} e^{\beta \tau} g(\tau) d\tau \quad \forall t \geq t_0. \quad (4.12)$$
(i) Case $\beta > \delta$. Setting $t_0 = 0$ in (4.12), we estimate

$$\|z(t)\| \leq e^{-\beta t}\|z(0)\| + e^{-\beta t}\int_0^t e^{\beta \tau}\|g(\tau)\|d\tau \leq e^{-\delta t}\|z(0)\| + e^{-\beta t}\int_0^t e^{\beta \tau}Me^{-\delta \tau}d\tau$$

$$= e^{-\delta t}\|z(0)\| + \frac{Me^{-\beta t}}{\beta - \delta}(e^{(\beta - \delta)t} - 1) \leq e^{-\delta t}\left(\|z(0)\| + \frac{M}{\beta - \delta}\right),$$

which implies (4.6).

(ii) Case $\beta = 0$. Note from (4.4) that $z(0) = y(0) - q(0) = -\int_0^\infty g(\tau)d\tau$. Letting $t_0 = 0$ in (4.11) gives

$$z(t) = z(0) + \int_0^t g(\tau)d\tau = -\int_t^\infty g(\tau)d\tau.$$ Hence

$$\|z(t)\| \leq \int_t^\infty \|g(\tau)\|d\tau \leq \int_t^\infty Me^{-\delta \tau}d\tau = \frac{M}{\delta}e^{-\delta t},$$

which proves (4.7).

(iii) Case $\beta < 0$. By (4.8) and the fact $q(t)$ is a polynomial, we have $e^{\beta t}z(t) \to 0$ as $t \to \infty$. Then letting $t \to \infty$ in (4.11) and setting $t_0 = t$ yield

$$z(t) = -e^{-\beta t}\int_t^\infty e^{\beta \tau}g(\tau)d\tau.$$ It follows that

$$\|z(t)\| \leq e^{-\beta t}\int_t^\infty Me^{(\beta - \delta)\tau}d\tau = e^{-\beta t}\frac{Me^{(\beta - \delta)t}}{-\beta + \delta} = \frac{M}{|\beta| + \delta}e^{-\beta t},$$

which proves (4.9). \qed

The remainder of this paper is focused on the proofs of main results, and will use the following notation.

**Notation.** If $n \in \sigma(A)$, we define $R_n$ to be the orthogonal projection in $H$ on the eigenspace of $A$ corresponding to $n$. In case $n \notin \sigma(A)$, set $R_n = 0$.

For $n \in \mathbb{N}$, define $P_n = R_1 + R_2 + \cdots + R_n$. Note that each vector space $P_nH$ is finite dimensional.

### 4.1 Proof of Theorem 2.2

We start by obtaining some additional properties for the force $f(t)$ and solution $u(t)$ which we will make use of later.

By the expansion (2.8) of $f(t)$ in $E_{\infty,\sigma_0}$, for each $N \in \mathbb{N}$ and $\alpha \geq 0$, there exists a number $\delta_{N,\alpha} \in (0, 1)$ such that

$$\left|f(t) - \sum_{n=1}^N f_n(t)e^{-nt}\right|_{\alpha,\sigma_0} = O(e^{-(N + \delta_{N,\alpha})t}) \quad \text{as } t \to \infty. \quad (4.13)$$

Observe that we have the following immediate consequences:
(a) The relation (4.13) implies for each \( \alpha \geq 0 \) that \( f(t) \) belongs to \( G_{\alpha,\sigma_0} \) for \( t \) large.

(b) Note that when \( N = 1 \), the function \( f(t) \) itself satisfies

\[
|f(t) - f_1(t)e^{-t}|_{\alpha,\sigma_0} = \mathcal{O}(e^{-(1+\delta_1,\alpha)t}).
\]

Since \( f_1(t) \) is a polynomial, it follows that

\[
|f(t)|_{\alpha,\sigma_0} = \mathcal{O}(e^{-\lambda t}), \quad \forall \lambda \in (0, 1), \forall \alpha \geq 0.
\]  (4.14)

Consequently, for any \( \varepsilon > 0 \), \( \alpha \geq 0 \), and \( \lambda \in (0, 1) \), applying (4.14) with \((\lambda + 1)/2\) replacing \( \lambda \), it follows that there is \( T > 0 \) such that

\[
|f(T + t)|_{\alpha,\sigma_0} \leq \varepsilon e^{-\lambda t} \quad \forall t \geq 0.
\]  (4.15)

(c) Combining (4.14) for \( \alpha = 0 \), with the Basic Assumption, we assume, without loss of generality, for each \( \lambda \in (0, 1) \) that

\[
|f(t)| \leq M_\lambda e^{-\lambda t}, \quad \forall t \geq 0, \text{ for some } M_\lambda > 0.
\]  (4.16)

For the solution \( u(t) \), we summarize the key estimates in section 3 into the following.

Claim. For any \( \alpha \geq 0 \) and \( \delta \in (0, 1) \), there exists a positive number \( T_* > 0 \) such that \( u(t) \) is a regular solution on \([T_*, \infty)\), and one has for \( t \geq 0 \) that

\[
|u(T_* + t)|_{\alpha+1/2,\sigma_0} \leq e^{-(1-\delta)t},
\]  (4.17)

\[
|B(u(T_* + t), u(T_* + t))|_{\alpha,\sigma_0} \leq e^{-2(1-\delta)t}.
\]  (4.18)

Proof of Claim. We apply Proposition 3.3. By (4.15), we have that (3.19) holds for \( \sigma = \sigma_0, \) any \( \alpha \geq 1/2 \) and any \( \lambda_0 \in (0, 1). \) Also, (3.15) is satisfied because of (4.16). Therefore (3.20) and (3.21) hold for \( \sigma = \sigma_0, \) any \( \alpha \geq 1/2 \) and any \( \delta \in (0, 1). \) These directly yield (4.17) and (4.18).

Returning to the main proof, it suffices to prove that there exist polynomials \( q_n \)'s for all \( n \geq 1 \) such that for each \( N \geq 1 \) the following properties \((\mathcal{H}1), (\mathcal{H}2), \) and \((\mathcal{H}3)\) hold true:

\( (\mathcal{H}1) \) \( q_N \in \mathcal{D}^{\infty,\sigma_0}. \)

\( (\mathcal{H}2) \) For \( \alpha \geq 1/2, \)

\[
\left|u(t) - \sum_{n=1}^{N} q_n(t)e^{-nt}\right|_{\alpha,\sigma_0} = \mathcal{O}(e^{-(N+\varepsilon)t}) \quad \text{as } t \to \infty, \quad \forall \varepsilon \in (0, \delta_{N,\alpha}^*),
\]  (4.19)

where the numbers \( \delta_{n,\alpha}^* \)'s, for \( \alpha \geq 1/2, \) are defined recursively by

\[
\delta_{n,\alpha}^* = \begin{cases} \delta_{1,\alpha}, & \text{for } n = 1, \\ \min\{\delta_{n,\alpha}, \delta_{n-1,\alpha+1/2}^*\}, & \text{for } n \geq 2. 
\end{cases}
\]
(H3) The ODE (2.11) holds in $E^{∞,σ_0}$ for $n = N$.

We prove these statements by constructing the polynomials $q_N(t)$’s recursively.

**Base case:** $N = 1$. Let $k ≥ 1$. By taking $v ∈ R_kH$ in the the weak formulation (2.6), we have

$$\frac{d}{dt} R_ku + kR_ku = R_k(\alpha(t) - B(u(t), u(t)))$$

(4.20) in the $R_kH$-valued distribution sense on $(0,∞)$. (Since $R_kH$ is finite dimensional, “$R_kH$-valued distribution sense,” is simply the same as “distribution sense”.)

Let $w_0(t) = e^tu(t)$ and $w_0(t) = R_kw_0(t)$. By virtue of the $H_w$-continuity of $u(t)$ (see (2.5)), we have $w_{0,k} ∈ C([0,∞), R_kH)$. It follows from (4.20) that

$$\frac{d}{dt} w_{0,k} + (k - 1)w_{0,k} = R_kf_1 + R_kH_0(t),$$

(4.21)

where

$$H_0(t) = e^t(\alpha(t) - F_1(t) - B(u(t), u(t))),$$

(4.22)

and $F_1$ is defined in (2.10). Note that $R_kf_1(t)$ is an $R_kH$-valued polynomial in $t$.

Let $α ≥ 1/2$ be fixed. Using (4.13) for $N = 1$ and applying (4.18) with $δ = (1 - δ_{1,α})/4$, there are $T_0 > 0$ and $D_0 ≥ 1$ such that for $t ≥ 0$,

$$e^t|\alpha(T_0 + t)|α,σ_0 ≤ D_0e^{-δ_{1,α}t},$$

(4.23)

$$e^t|\alpha(T_0 + t)|α,σ_0 ≤ e^{(2δ - 1)t} = e^{-(1+δ_{1,α})t}/2 ≤ e^{-δ_{1,α}t}.$$  

(4.24)

Then, by setting $D_1 = D_0 + 1$, we have

$$|H_0(T_0 + t)|_α,σ_0 ≤ D_1e^{-δ_{1,α}t}, \ \ ∀t ≥ 0.$$  

(4.25)

We will now identify the components of the desired polynomial, $q_1(t)$, belonging to each eigenspace $R_kH$.

**Case: $k = 1$.** Applying Lemma (4.2) ii) to equation (4.21) with $X = R_1H$, $|| · || = | · |_α,σ_0$,

$$y(t) = w_{0,1}(T_0 + t), \ \ p(t) = R_1f_1(T_0 + t), \ \ g(t) = R_1H_0(T_0 + t),$$

we infer that there is an $R_1H$-valued polynomial $q_{1,1}(t)$ such that for any $t ≥ 0$

$$|w_{0,1}(T_0 + t) - q_{1,1}(t)|_α,σ_0 ≤ \frac{D_1}{δ_{1,α}}e^{-δ_{1,α}t},$$

thus,

$$|R_1w_0(t) - q_{1,1}(t - T_0)|_{α,σ_0} ≤ \frac{D_1e^{δ_{1,α}T_0}}{δ_{1,α}}e^{-δ_{1,α}t}, \ \ ∀t ≥ T_0.$$  

(4.26)

In fact,

$$q_{1,1}(t) = ξ_1 + \int_0^t R_1f_1(τ + T_0)dτ \ \ \text{for some} \ ξ_1 ∈ R_1H.$$  

(4.27)
Case: $k \geq 2$. We apply Lemma 4.2(i) to equation (4.21) with
\[ y(t) = w_{0,k}(T_0 + t), \quad p(t) = R_k f_1(T_0 + t), \quad g(t) = R_k H_0(T_0 + t), \]
where $\beta = k - 1 > \delta_{1,\alpha}$, and the norm $\| \cdot \|$ being $| \cdot |_{\alpha,\sigma_0}$ on the space $X = R_k H$. In particular, there is an $R_k H$-valued polynomial, $q_{1,k}(t)$ such that for any $t \geq 0$
\[ |w_{0,k}(T_0 + t) - q_{1,k}(t)|_{\alpha,\sigma_0} \leq e^{-\delta_{1,\alpha}t} \left( |w_{0,k}(T_0)|_{\alpha,\sigma_0} + |q_{1,k}(0)|_{\alpha,\sigma_0} + \frac{D_1}{k - 1 - \delta_{1,\alpha}} \right), \]
which implies for all $t \geq T_0$ that
\[ |w_{0,k}(t) - q_{1,k}(t - T_0)|_{\alpha,\sigma_0} \leq e^{-\delta_{1,\alpha}(t-T_0)} \left( |w_{0,k}(T_0)|_{\alpha,\sigma_0} + |q_{1,k}(0)|_{\alpha,\sigma_0} + \frac{D_1}{k - 1 - \delta_{1,\alpha}} \right). \quad (4.28) \]
In fact, for $k \geq 2$
\[ q_{1,k}(t) = -e^{(1-k)t} \int_{-\infty}^{t} e^{(k-1)\tau} R_k f_1(T_0 + \tau) d\tau. \quad (4.29) \]

**Polynomial $q_1(t)$**. Define
\[ q_1(t) = \sum_{k=1}^{\infty} q_{1,k}(t - T_0), \quad t \in \mathbb{R}. \quad (4.30) \]
We next prove that $q_1 \in \mathcal{P}^{\infty,\sigma_0}$. Write
\[ f_1(t + T_0) = \sum_{d=0}^{m} a_d t^d, \quad \text{for some } a_d \in E^{\infty,\sigma_0}. \]
Clearly, by (4.27), $R_1 q_1(t + T_0) = q_{1,1}(t)$ is a $\mathcal{V}$-valued polynomial, and hence,
\[ \text{the mapping } t \mapsto R_1 q_1(t + T_0) \text{ belongs to } \mathcal{P}^{\infty,\sigma_0}. \quad (4.31) \]
We consider the remaining part $(I - R_1) q_1(t + T_0)$. Using the integral formula (4.1),
\begin{align*}
(I - R_1) q_1(t + T_0) &= \sum_{k=2}^{\infty} q_{1,k}(t) = \sum_{k=2}^{\infty} -e^{(1-k)t} \int_{-\infty}^{t} e^{(k-1)\tau} \left( \sum_{d=0}^{m} R_k a_d \tau^d \right) d\tau \\
&= \sum_{k=2}^{\infty} \frac{1}{k - 1} \sum_{d=0}^{m} \sum_{n=0}^{d} \frac{(-1)^{d-n} d!}{n!(k-1)^{d-n} R_k a_d} t^n \\
&= \sum_{k=2}^{\infty} \frac{1}{k - 1} \sum_{n=0}^{d} \left( \sum_{d=0}^{m} \frac{(-1)^{d-n} d!}{n!(k-1)^{d-n}} R_k a_d \right) t^n.
\end{align*}
Thus,
\[ (I - R_1) q_1(t + T_0) = \sum_{n=0}^{d} b_n t^n, \quad (4.32) \]
where the coefficient $b_n$, for $0 \leq n \leq d$, is

$$b_n = \sum_{k=2}^{\infty} \frac{1}{k-1} \left( \sum_{d=n}^{m} \frac{(-1)^{d-n} d!}{n!(k-1)^{d-n}} R_k a_d \right).$$

For any $\mu \geq 0$, we have

$$|b_n|_{\mu+1,\sigma_0}^2 = |A b_n|_{\mu,\sigma_0}^2 = \sum_{k=2}^{\infty} \frac{1}{k-1} \sum_{d=n}^{m} \frac{(-1)^{d-n} d!}{n!(k-1)^{d-n}} \cdot |R_k a_d|_{\mu,\sigma_0}^2 \leq \sum_{k=2}^{\infty} \frac{k^2}{(k-1)^2} \left( \sum_{d=n}^{m} \frac{m!}{n!} |R_k a_d|_{\mu,\sigma_0} \right)^2 \leq 4 \sum_{k=2}^{\infty} \frac{(m!)^2(m - n + 1)^2}{(n!)^2} |R_k a_d|_{\mu,\sigma_0}^2.$$

(Above, we simply used $k/(k-1) \leq 2$ for the last inequality.) Thus,

$$|b_n|_{\mu+1,\sigma_0}^2 \leq \frac{4(m!)^2(m - n + 1)^2}{(n!)^2} |(I - R_1) a_d|_{\mu,\sigma_0}^2 < \infty.$$

Therefore, $b_n \in E^{\infty,\sigma_0}$ for all $0 \leq n \leq d$, and, by (4.32), the mapping $t \mapsto (I - R_1) q_1(t + T_0)$ belongs to $\mathcal{P}^{\infty,\sigma_0}$. This, together with (4.31), implies that $t \mapsto q_1(t + T_0)$ belongs to $\mathcal{P}^{\infty,\sigma_0}$ and, ultimately, that $t \mapsto q_1(t)$ belongs to $\mathcal{P}^{\infty,\sigma_0}$, as desired.

**Remainder estimate.** We estimate $|u(t) - q_1(t)e^{-t}|_{\alpha,\sigma_0}$ now. Firstly, inequality (4.26) yields

$$|R_1(w_0(t) - q_1(t))|_{\alpha,\sigma_0} = O(e^{-\delta_1,\alpha t}).$$

Secondly, we have from (4.28) that

$$\sum_{k=2}^{\infty} |R_k(w_0(t) - q_1(t))|_{\alpha,\sigma_0} \leq 3e^{2\delta_1,\alpha T_0} e^{-2\delta_1,\alpha t} \sum_{k=2}^{\infty} \left( |w_0(kT_0)|_{\alpha,\sigma_0}^2 + |R_k q_1(T_0)|_{\alpha,\sigma_0}^2 + \frac{D_1^2}{(k - 1 - \delta_1,\alpha)^2} \right) \leq D_2^2 e^{-2\delta_1,\alpha t}$$

for all $t \geq T_0$, where

$$D_2^2 = 3e^{2\delta_1,\alpha T_0} \left\{ |(I - R_1) w_0(T_0)|_{\alpha,\sigma_0}^2 + |(I - R_1) q_1(T_0)|_{\alpha,\sigma_0}^2 + D_1^2 \sum_{k=2}^{\infty} \frac{1}{(k - 1 - \delta_1,\alpha)^2} \right\} < \infty.$$

This implies

$$|(I - R_1)(w_0(t) - q_1(t))|_{\alpha,\sigma_0} \leq D_2 e^{-\delta_1,\alpha t}, \quad \forall t \geq T_0.$$

Combining (4.34) with (4.35) gives

$$|w_0(t) - q_1(t)|_{\alpha,\sigma_0} = O(e^{-\delta_1,\alpha t}),$$
and consequently,
\[ |u(t) - q_1(t)e^{-t}|_{\alpha,\sigma_0} = O(e^{-(1+\delta_1,\alpha)t}). \] (4.36)

Thanks to (4.36), the polynomial \( q_1(t) \) is independent of \( \alpha \). Hence the same \( q_1 \) satisfies (4.36) for all \( \alpha \geq 1/2 \), which proves (H2) for \( N = 1 \).

This proves (4.19) for \( N = 1 \).

**Establishing the ODE (2.12).** By (4.36) in Lemma 4.2, the polynomial \( q_1(t) \) satisfies
\[ \frac{d}{dt} R_k q_1(T_0 + t) + (k - 1) R_k q_1(T_0 + t) = R_k f_1(T_0 + t), \quad \forall k \geq 1, \quad \forall t \in \mathbb{R}. \] (4.37)

For each \( \mu \geq 0 \), we have \( A q_1(T_0 + t) \) and \( f_1(T_0 + t) \) belong to \( G_{\mu,\sigma_0} \). Hence, we can sum over \( k \) in (4.37) and obtain
\[ \frac{d}{dt} q_1(t) + (A - 1) q_1(t) = f_1(t) \quad \text{in} \ G_{\mu,\sigma_0}, \quad \forall t \in \mathbb{R}, \]

which implies that the differential equation (2.12) holds in \( E^{2,\sigma_0} \).

Therefore, \( q_1 \) satisfies (H1), (H2), and (H3) for \( N = 1 \).

**Recursive step.** Let \( N \geq 1 \). Suppose that there already exist \( q_1, q_2, \ldots, q_N \in P^{\infty,\sigma_0} \) that satisfy (H2), and the ODE (2.11) holds in \( E^{\infty,\sigma_0} \) for each \( n = 1, 2, \ldots, N \).

We will construct a polynomial \( q_{N+1}(t) \) that satisfies (H1), (H2), and (H3) with \( N + 1 \) replacing \( N \).

Let \( \alpha \geq 1/2 \) be given and \( \varepsilon_* \) be arbitrary in \( (0, \delta_{N+1,\alpha}) \). Define
\[ \bar{u}_N = \sum_{n=1}^{N} u_n \quad \text{and} \quad v_N = u - \bar{u}_N. \]

Assumption (H2) particularly yields
\[ |v_N(t)|_{\alpha+1/2,\sigma_0} = O(e^{-(N+\varepsilon_*)t}), \quad \forall \varepsilon_* \in (0, \delta_{N,\alpha+1/2}). \] (4.38)

Subtracting (2.11) for \( n = 1, 2, \ldots, N \) from (2.2), we have
\[ \frac{d}{dt} v_N + A v_N + B(u, u) - \sum_{m+j \leq N} B(u_m, u_j) = f - \sum_{n=1}^{N} F_n, \] (4.39)

where the functions \( F_n \)'s are defined in (2.10). We reformulate equation (4.39) as
\[ \frac{d}{dt} v_N + A v_N + \sum_{m+j = N+1} B(u_m, u_j) = F_{N+1} + h_N, \] (4.40)

where
\[
\begin{align*}
h_N &= -B(u, u) + \sum_{1 \leq m, j \leq N \atop m+j \leq N+1} B(u_m, u_j) + f - \sum_{n=1}^{N+1} F_n \\
&= -\left\{ B(u, u) - B(\bar{u}_N, \bar{u}_N) \right\} - \left\{ B(\bar{u}_N, \bar{u}_N) - \sum_{1 \leq m, j \leq N \atop m+j \leq N+1} B(u_m, u_j) \right\} + \left\{ f - \sum_{n=1}^{N+1} F_n \right\}.
\end{align*}
\]
With this way of grouping, we rewrite $h_N$ as

$$h_N = -B(v_N, u) - B(\tilde{u}_N, v_N) - \sum_{1 \leq m, j \leq N, m+j \geq N+2} B(u_m, u_j) + \tilde{F}_{N+1}, \quad (4.41)$$

where

$$\tilde{F}_{N+1}(t) = f(t) - \sum_{n=1}^{N+1} F_n(t).$$

Note in case $N = 1$ that neither of the following terms

$$\sum_{m+j \leq N} B(u_m, u_j) \text{ in (4.39) nor} \sum_{1 \leq m, j \leq N, m+j \geq N+2} B(u_m, u_j) \text{ in (4.41)}$$

will appear.

**Estimate of $h_N(t)$.** By (4.13) and Remark 2.3 we have

$$|\tilde{F}_{N+1}(t)|_{\alpha, \sigma_0} = O(e^{-(N+1+\delta_{N+1, \alpha})t}) = O(e^{-(N+1+\varepsilon_*)t}). \quad (4.42)$$

It is also obvious that

$$\sum_{1 \leq m, j \leq N, m+j \geq N+2} |B(u_m, u_j)|_{\alpha, \sigma_0} = \sum_{1 \leq m, j \leq N, m+j \geq N+2} e^{-(m+j)t} |B(q_m, q_j)|_{\alpha, \sigma_0} = O(e^{-(N+1+\varepsilon_*)t}). \quad (4.43)$$

Take $\varepsilon \in (\varepsilon_*, \delta_{N+1, \alpha}) \subset (0, \delta_{N+1/2, \alpha})$ in (4.38), and set $\delta = \varepsilon - \varepsilon_* \in (0, 1)$. Then we have from the definition of $u_n(t)$ and (4.17) that

$$|\tilde{u}_N(t)|_{\alpha+1/2, \sigma_0}, |u(t)|_{\alpha+1/2, \sigma_0} = O(e^{-(1-\delta)t}). \quad (4.44)$$

By Lemma 3.1 and estimates (4.38), (4.44), it follows that

$$|B(v_N, u)|_{\alpha, \sigma_0}, |B(\tilde{u}_N, v_N)|_{\alpha, \sigma_0} = O(e^{-(N+\varepsilon+1-\delta)t}) = O(e^{-(N+1+\varepsilon_*)t}). \quad (4.45)$$

Therefore, by (4.41), (4.42), (4.43) and (4.45),

$$|h_N(t)|_{\alpha, \sigma_0} = O(e^{-(N+1+\varepsilon_*)t}). \quad (4.46)$$

**Construction of $q_{N+1}(t)$.** Using the weak formulation of (4.40), which is similar to (2.6), and then taking the test function, $v$, to be in $R_kH$, we obtain

$$\frac{d}{dt} R_k v_N + k R_k v_N + \sum_{m+j = N+1} R_k B(u_m, u_j) = R_k F_{N+1} + R_k h_N \text{ on } (0, \infty). \quad (4.47)$$

Let $w_N(t) = e^{(N+1)t} v_N(t)$ and $w_{N,k} = R_k w_N(t)$. Using (4.47), we write the ODE for $w_{N,k}$ as

$$\frac{d}{dt} w_{N,k} + (k - (N+1)) w_{N,k} = \left( R_k f_{N+1} - \sum_{m+j = N+1} R_k B(q_m, q_j) \right) + H_{N,k}, \quad (4.48)$$
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with $H_{N,k}(t) = e^{(N+1)t}R_kh_N(t)$. Note from (4.46) that

$$|H_{N,k}(t)|_{\alpha,\sigma_0} = \mathcal{O}(e^{-\varepsilon_\star t}).$$

Then there exist $T_N > 0$ and $D_3 > 0$ such that

$$|H_{N,k}(T_N + t)|_{\alpha,\sigma_0} \leq D_3 e^{-\varepsilon_\star t}, \quad \forall t \geq 0.$$

By the first property in (2.5), each $w_{N,k}(t)$ is continuous from $[0, \infty)$ to $R_kH$.

Case $k = N + 1$. By Lemma 4.2(ii) applied to equation (4.48) on $(T_N, \infty)$, there is a polynomial $q_{N+1,N+1}(t)$ valued in $R_{N+1}H$ such that

$$|w_{N,N+1}(T_N + t) - q_{N+1,N+1}(t)|_{\alpha,\sigma_0} = \mathcal{O}(e^{-\varepsilon_\star t}).$$

Thus,

$$|R_{N+1}w_N(t) - q_{N+1,N+1}(t - T_N)|_{\alpha,\sigma_0} = \mathcal{O}(e^{-\varepsilon_\star t}). \quad (4.49)$$

Case $k \leq N$. Note that by (4.38)

$$\lim_{t \to \infty} e^{(k-(N+1))t}w_{N,k}(t) = \lim_{t \to \infty} e^{kt}R_kv_N(t) = 0.$$

By applying Lemma 4.2(iii) to equation (4.48) on $(T_N, \infty)$ with $\beta = k - N - 1 < 0$, there is a polynomial $q_{N+1,k}(t)$ valued in $R_kH$ such that

$$|w_{N,k}(T_N + t) - q_{N+1,k}(t)|_{\alpha,\sigma_0} = \mathcal{O}(e^{-\varepsilon_\star t}),$$

hence,

$$|R_kw_N(t) - q_{N+1,k}(t - T_N)|_{\alpha,\sigma_0} = \mathcal{O}(e^{-\varepsilon_\star t}). \quad (4.50)$$

Case $k \geq N + 2$. Similarly, applying Lemma 4.2(i) to equation (4.48) on $(T_N, \infty)$ with $\beta = k - N - 1 > \varepsilon_\star$, there is a polynomial $q_{N+1,k}(t)$ valued in $R_kH$ such that for $t \geq 0$,

$$|w_{N,k}(T_N + t) - q_{N+1,k}(t)|_{\alpha,\sigma_0} \leq \left(|R_kv_N(T_N)|_{\alpha,\sigma_0} + |q_{N+1,k}(0)|_{\alpha,\sigma_0} + \frac{M}{k - (N+1)}\right)e^{-\varepsilon_\star t}.$$

Thus

$$|R_kw_N(t) - q_{N+1,k}(t - T_N)|_{\alpha,\sigma_0} \leq e^{\varepsilon_\star T_N} \left(|R_kv_N(T_N)|_{\alpha,\sigma_0} + |q_{N+1,k}(0)|_{\alpha,\sigma_0} + \frac{M}{k - (N+1)}\right)e^{-\varepsilon_\star t} \quad \forall t \geq T_N. \quad (4.51)$$

We define

$$q_{N+1}(t) = \sum_{k=1}^{\infty} q_{N+1,k}(t - T_N), \quad t \in \mathbb{R}.$$

It follows from (2.8) that $f_{N+1} \in \mathcal{P}^{\infty,\sigma_0}$, and from the recursive assumptions that $q_m, q_j \in \mathcal{P}^{\infty,\sigma_0}$ for $1 \leq m, j \leq N$; then by (3.4), we have $f_{N+1} - \sum_{m+j=N+1} B(q_m, q_j) \in \mathcal{P}^{\infty,\sigma_0}$. 
Following the same proof that shows \( q_1 \in P^{\infty,\sigma_0} \), one can argue similarly for \( q_{N+1} \) to have \( q_{N+1} \in P^{\infty,\sigma_0} \).

**Estimate of** \( v_{N+1}(t) \). From (1.49) and (1.50), we immediately have

\[
|P_{N+1}(w_N(t) - q_{N+1}(t))|_{\alpha,\sigma_0} = O(e^{-\varepsilon_* t}).
\] (4.52)

Squaring (4.51) and summing over \( k \geq N + 2 \), we obtain for \( t \geq T_N \) that

\[
|(I - P_{N+1})(w_N(t) - q_{N+1}(t))|_{\alpha,\sigma_0}^2 = \sum_{k=N+2}^{\infty} |R_k(w_N(t) - q_{N+1}(t))|_{\alpha,\sigma_0}^2
\]

\[
\leq 3e^{2\varepsilon_* T_N} \left( \sum_{k=N+2}^{\infty} |R_k v_N(T_N)|_{\alpha,\sigma_0}^2 + \sum_{k=N+2}^{\infty} |R_k q_{N+1}(T_N)|_{\alpha,\sigma_0}^2 + \sum_{k=N+2}^{\infty} \frac{M^2}{(k - (N + 1))^2} \right) e^{-2\varepsilon_* t}.
\]

Since the last three sums are convergent, we deduce

\[
|(I - P_{N+1})(w_N(t) - q_{N+1}(t))|_{\alpha,\sigma_0} = O(e^{-\varepsilon_* t}).
\] (4.53)

From (4.52) and (4.53), we have

\[
|w_N(t) - q_{N+1}(t)|_{\alpha,\sigma_0} = O(e^{-\varepsilon_* t}).
\] (4.54)

Therefore,

\[
|v_{N+1}(t)|_{\alpha,\sigma_0} = |v_N(t) - e^{-(N+1)t} q_{N+1}(t)|_{\alpha,\sigma_0} = O(e^{-(N+1+\varepsilon_*)t}).
\] (4.55)

Thanks to (4.54), the polynomial \( q_{N+1}(t) \) is independent of \( \alpha \) and \( \varepsilon_* \). Therefore, (4.55) holds for any \( \alpha \geq 1/2 \) and \( \varepsilon_* \in (0, \delta^*_{N+1}) \), which proves (H2) with \( N + 1 \) replacing \( N \).

**Establishing the ODE** (2.11) **for** \( n = N + 1 \). By our construction of the polynomials \( q_{N+1,k}(t) \) above, and by (4.50) in Lemma 4.2, the polynomial \( q_{N+1}(t) \) satisfies

\[
\frac{d}{dt} R_k q_{N+1}(T_N + t) + (k - (N + 1)) R_k q_{N+1}(T_N + t)
\]

\[
+ \sum_{m+j=N+1} R_k B(q_m(T_N + t), q_j(T_N + t)) = R_k f_{N+1}(T_N + t) \quad \forall k \geq 1, \quad \forall t \in \mathbb{R}.
\]

This yields, for each \( k \geq 1 \),

\[
\frac{d}{dt} R_k u_{N+1}(t) + AR_k u_{N+1}(t) + \sum_{m+j=N+1} R_k B(u_m(t), u_j(t)) = R_k F_{N+1}(t), \quad \forall t \in \mathbb{R}.
\] (4.56)

For any \( \mu \geq 0 \), since \( A u_{N+1}(t), \sum_{m+j=N+1} B(u_m(t), u_j(t)), F_{N+1}(t) \) each is a \( G_{\mu,\sigma_0} \)-valued polynomial, then summing up equation (4.56) in \( k \) gives

\[
\frac{d}{dt} u_{N+1} + Au_{N+1} + \sum_{m+j=N+1} B(u_m, u_j) = F_{N+1} \quad \text{in } G_{\mu,\sigma_0}.
\]

Thus, the ODE (2.11) holds in \( E^{\infty,\sigma_0} \) for \( n = N + 1 \).

We have established the existence of the desired polynomial, \( q_{N+1}(t) \), which completes the recursive step, and hence, the proof of Theorem 2.2. \( \square \)

**Remark 4.3.** By using the extra information (2.13) in Remark 2.3, we can prove directly the remainder estimate (1.19) for any \( \varepsilon \in (0,1) \), which, in fact, is expected by (2.14). Nonetheless, the above proof with specific \( \varepsilon \in (0, \delta^*_{N+1}) \) is more flexible and will be easily adapted in section 4.3 below to serve the proof of Theorem 2.6.
4.2 Proof of Corollary 2.5

We follow the proof of Theorem 2.2. Since $f_1 \in \mathcal{V}$, there is $N_1 \geq 1$ such that $f_1 \in P_{N_1}H$. As a consequence, we see from (4.29) that $q_{1,k} = 0$ for $k > N_1$. Hence, $q_1(t) = \sum_{k=1}^{N_1} q_{1,k}(t - T_0)$ is a polynomial in $P_{N_1}H$.

For the recursive step, the functions $f_{N+1}$, $g_m$ and $q_j$ $(1 \leq m, j \leq N)$, in this case, are $\mathcal{V}$-valued polynomials. Hence, by (2.1), so is $f_{N+1} = \sum_{m+j=N+1} B(g_m, q_j)$. It follows that there are at most finitely many $k$ such that $q_{N+1,k}$ is nonzero. Since each $q_{N+1,k}$ is an $\mathcal{V}$-valued polynomial, clearly $q_{N+1}$, as a finite sum of those, is a $\mathcal{V}$-valued polynomial. □

4.3 Proof of Theorem 2.6

We follow the proof of Theorem 2.2 closely and make necessary modifications. We prove part (i), while the proof of part (ii) is similar and omitted.

The same notation $u_n(t)$, $F_n(t)$, $v_n(t)$, $\bar{u}_n(t)$ as in Theorem 2.2 is used here.

By (2.15) with $N = 1$,

$$e^t |f(t) - F_1(t)|_{\mathcal{V}} = O(e^{-\delta_1 \cdot t}).$$

(4.57)

Also,

$$|f(t)|_{\mathcal{V}} \leq |f_1(t)|_{\mathcal{V}} e^{-t} + |f(t) - f_1(t) e^{-t}|_{\mathcal{V}} = O(e^{-\lambda t}), \quad \forall \lambda \in (0, 1).$$

(4.58)

Using (4.58) and by applying Proposition 3.3 we have

$$|u(t)|_{\mathcal{V}} = O(e^{-(1-\delta)t}), \quad \forall \delta \in (0, 1),$$

(4.59)

and

$$|B(u(t), u(t))|_{\mathcal{V}} = O(e^{-2(1-\delta)t}), \quad \forall \delta \in (0, 1).$$

(4.60)

**Base case:** $N = 1$. Let $\alpha = \alpha_*$ and $\mu = \mu_*$.

In estimating $H_0(t)$ defined by (4.22), the estimate (4.23), resp. (4.24), comes from (4.57), resp. (4.60). Hence we obtain the bound (4.25) for $|H_0(t)|_{\mathcal{V}}$. Then the existence and definition of $q_1(t)$ are the same as in (4.27), (4.29) and (4.30).

Since $f_1 \in \mathcal{P}^{\mu_\ast, \sigma_0}$, then the same proof gives $q_1 \in \mathcal{P}^{\mu_\ast, \sigma_0}$, see (4.31), (4.32) and (4.33). The remainder estimate (4.36) still holds true, which, for the current value $\alpha = \alpha_*$, proves (2.16) for $N = 1$. Also, the ODE (2.12) holds in $G_{\mu_\ast, \sigma_0}$ (for the current $\mu = \mu_*$).

If $N_\ast = 1$, then the proof is finished here. We consider $N_\ast \geq 2$ now.

**Recursive step.** Let $1 \leq N \leq N_\ast - 1$. Assume there already exist $q_n \in \mathcal{P}^{\mu_n, \sigma_0}$ for $1 \leq n \leq N$, such that

$$|v_N(t)|_{G_{\mu_N, \sigma_0}} = O(e^{-(N+\varepsilon)t}), \quad \forall \varepsilon \in (0, \delta^*_N),$$

(4.61)

and (2.11) holds in $G_{\mu_N, \sigma_0}$ for $n = 1, 2, 3, \ldots, N$.

Let

$$\alpha = \alpha_{N+1} = \alpha_N - 1/2 \geq 1/2, \quad \mu = \mu_{N+1} = \mu_N - 1/2 \geq \alpha \geq 1/2.$$

(4.62)

Note for $n = 1, 2, \ldots, N$ that $\mu_n \geq \alpha_n \geq 1/2$ and both $\mu_n$, $\alpha_n$ are decreasing, hence,

$$u_n(t), q_n(t) \in G_{\mu_n, \sigma_0} \subset G_{\mu_N, \sigma_0} = G_{\mu+1/2, \sigma_0} \subset G_{\alpha_N, \sigma_0} = G_{\alpha+1/2, \sigma_0}, \quad \forall t \in \mathbb{R}.$$
Rewrite (4.61) as
\[ |v_N(t)|_{\alpha+1/2,\sigma_0} = \mathcal{O}(e^{-(N+\varepsilon)t}), \quad \forall \varepsilon \in (0, \delta^*_N). \] (4.64)

We now construct a polynomial \( q_{N+1} \in \mathcal{P}^{\mu+1,\sigma_0} \) such that (2.16) holds true with \( N+1 \) replacing \( N \), and the ODE (2.11), with \( n = N + 1 \), holds in \( G_{\mu,N+1,\sigma_0} = G_{\mu,\sigma_0} \).

We proceed with the construction of \( q_{N+1}(t) \) as in the proof of Theorem 2.2 using the specific values of \( \alpha \) and \( \mu \) in (4.62).

Note that equation (4.40) for \( v_N \) still holds in the weak sense as in Definition 2.1.

- We check the estimate (4.46) for the function \( h_N(t) \) defined by (4.41).
- Let \( \varepsilon_* \in (0, \delta^*_N) \). By (2.15) with \( N + 1 \) replacing \( N \), we have
  \[ |\hat{F}_{N+1}(t)|_{\alpha,\sigma_0} = \mathcal{O}(e^{-(N+1+\delta^*_N+\varepsilon_*)t}) = \mathcal{O}(e^{-(N+1+\varepsilon_*)t}). \] (4.65)

Thanks to (4.63) and Lemma 3.1, estimate (4.43) stays the same as
  \[ \sum_{1 \leq m, j \leq N \atop m+j \geq N+2} |B(u_m, u_j)|_{\alpha,\sigma_0} = \mathcal{O}(e^{-(N+1+\varepsilon_*)t}). \] (4.66)

Again, take \( \varepsilon \in (\varepsilon_*, \delta^*_N) \subset (0, \delta^*_N) \) in (4.64) and \( \delta = \varepsilon - \varepsilon_* \in (0, 1) \). Then we have
  \[ |\bar{u}_N(t)|_{\alpha+1/2,\sigma_0} = |\bar{u}_N(t)|_{\alpha,\sigma_0} = \mathcal{O}(e^{-(1-\delta)t}), \] (4.67)

and by (4.59)
  \[ |u(t)|_{\alpha+1/2,\sigma_0} \leq |u(t)|_{\alpha_*,\sigma_0} = \mathcal{O}(e^{-(1-\delta)t}). \]

By Lemma 3.1 and estimates (4.64), (4.67), it follows that
  \[ |B(v_N, u)|_{\alpha,\sigma_0}, |B(\bar{u}_N, v_N)|_{\alpha,\sigma_0} = \mathcal{O}(e^{-(N+\varepsilon+1-\delta)t}) = \mathcal{O}(e^{-(N+1+\varepsilon_*)t}). \]

Therefore, by (4.65), (4.66) and (4.67), we again obtain (4.46).

- The same construction of \( q_{N+1}(t) \) now goes through.
- Since \( f_{N+1} \in \mathcal{P}^{\mu,\sigma_0} \), and, by (4.63), \( q_m, q_j \in \mathcal{P}^{\mu+1/2,\sigma_0} \) for \( 1 \leq m, j \leq N \), then by (3.3),
  \[ f_{N+1} - \sum_{m+j=N+1} B(q_m, q_j) \in \mathcal{P}^{\mu,\sigma_0}. \]

The same proof as for the case of \( q_1 \) yields that \( q_{N+1} \in \mathcal{P}^{\mu+1,\sigma_0} \).

- For the estimate of \( v_{N+1}(t) \), same arguments yield
  \[ |v_{N+1}(t)|_{\alpha,\sigma_0} = |v_N(t) - e^{-(N+1)t} q_{N+1}(t)|_{\alpha,\sigma_0} = \mathcal{O}(e^{-(N+1+\varepsilon_*)t}). \]

Since this holds for any \( \varepsilon_* \in (0, \delta^*_N) \), the remainder estimate (2.16) holds true with \( N + 1 \) replacing \( N \).

- As for the ODE (2.11) with \( n = N + 1 \), the proof is unchanged from that of Theorem 2.2, noting that the ODE now holds in the corresponding space \( G_{\mu,\sigma_0} \).

We have proved that the polynomial \( q_{N+1} \) has the desired properties. This completes the recursive step, and hence, the proof of Theorem 2.6. \( \square \)
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