AN ANALOGUE OF SPRINGER FIBERS IN CERTAIN WONDERFUL COMPACTIFICATIONS

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Abstract. We investigate the topological structure of a cellular decomposition of the fixed locus of a unipotent operator of regular Jordan type acting on the wonderful compactification of the variety of complete quadrics and the variety of complete skew forms. The Poincaré polynomial is computed in each case and the poset of cell closures under inclusion is described in the complete quadrics case.

1. Introduction

Let G denote the special linear group $\mathrm{SL}_n(\mathbb{C})$ of invertible $n \times n$ complex matrices with determinant 1. Let $\mathcal{U} \subset G$ denote the set of all unipotent elements, which forms an irreducible, closed subvariety of G. Consider the incidence variety $X = \{(gB, x) \in G/B \times G : \ g^{-1}xg \in \mathcal{U}\}$. It is well known that $\mathcal{U}$ is singular; however, the second projection $pr_2 : X \to G$ restricts to a resolution of singularities for $\mathcal{U}$ (see [17] and [21]). The fibers of pr_2, known as *Springer fibers*, have many remarkable combinatorial and geometric properties. For example, if $u \in G$ is a unipotent element, then for each $i \geq 0$ the cohomology space $H^i(pr_2^{-1}(u); \mathbb{Q})$ is a representation of the symmetric group S_n, and the top non-vanishing cohomology space is an irreducible S_n-module ([18], [19]). Furthermore, all irreducible components of $pr_2^{-1}(u)$ have the same dimension $e(u)$, which is equal to the half of the dimension of the top non-vanishing cohomology space $H^{2e(u)}(pr^{-1}(u); \mathbb{Q})$, [16]. These facts lead to striking combinatorial results as in [9, 12, 22].

Motivated by the combinatorial significance of the fibers of pr_2, in our earlier paper [2] we initiated a study of the sets

$$(G/K)_u := \{gK : \ g^{-1}ug \in K\}, \tag{1.1}$$

for fixed $u \in \mathcal{U}$, in the cases where $K = \mathrm{SO}_n$, the special orthogonal group, and $K = \mathrm{Sp}_n$, the symplectic group. Unlike the fibers $pr_2^{-1}(u)$, our sets $(G/K)_u$ are not complete. Therefore, in this paper we continue our analysis by studying their completions in a compactification of G/K. Since $u \cdot x = x$ is a closed condition, observe that the closure of $(G/K)_u$ in a completion $\overline{G/K}$ is equal to the u-fixed locus $(\overline{G/K})_u$ in the completion.

Suppose there exists an involution $\theta : G \to G$ such that $K = \{g \in G : \ \theta(g) = g\}$ is the fixed subgroup of θ. The two cases above fit this description with $K = \mathrm{SO}_n$ (resp. $K = \mathrm{Sp}_n$) being the fixed subgroup of $\theta(g) = (g^{-1})^\top$ (resp., $\theta(g) =$

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$J(g^{-1})^\top J^{-1}$ for an appropriate skew symmetric form J). The *wonderful embedding of* G/K is a canonical smooth projective G-variety $\mathcal{M} = \mathcal{M}_{G/K}$ that contains a copy of G/K as a G-stable open subset, and its boundary $\mathcal{M}\setminus G/K$ is a finite union of G-stable divisors with normal crossings [4]. The case where $G = G_1 \times G_1$, with G_1 reductive and $\theta(x, y) = (y, x)$, so that $K = \text{diag}(G_1)$ and $G/K \cong G_1$, has been studied previously. In fact, the wonderful embedding in this case is used in the study of character sheaves [13, 7]. The Zariski closures in the wonderful compactification of some closely related subvarieties of G_1 (called "Steinberg fibers") have also been studied [6, 8]. For a survey of these results, we recommend Springer's I.C.M. report [20].

Our main focus is on the fixed locus when the unipotent element u is regular, that is to say the Jordan type of u consists of a single block. Let X denote the wonderful compactification $\mathcal{M}_{G/H}$ as described above corresponding to $K = \text{SO}_n$ or $K = \text{Sp}_n$. Our first main result, Proposition 4.5 and Theorem 5.8, shows that X^u has a decomposition into affine cells associated to the flow of a one-parameter subgroup with finitely many fixed points. The explicit cellular decomposition allows us to compute the Poincaré polynomial of X^u in each of our two cases of interest in Corollaries 5.9 and 5.12. In the case of complete quadrics, we describe the poset of cell closure containments in Proposition 6.2 and compute a linear recurrence for the number of irreducible components in 6.11. Unlike the case of classical Springer fibers, we show through a simple example that the irreducible components of X^u are not equidimensional.

Our paper is organized as follows. After setting up our notational conventions in the Section 2, in Section 3.1 we recall some results from [2] that explicitly describe which matrices are invariant under the action $u \cdot A = (u^{-1})^\top A u^{-1}$ of a unipotent matrix u. In Section 3.2 we introduce one parameter subgroups that we use to describe the cell decompositions in Section 5. In Section 3.3, we give a brief overview of the construction of the wonderful compactifcation, focusing on the special cases of complete quadrics (Section 3.4) and complete skew forms (Section 3.5) in detail. In Section 4, we describe a certain decomposition of the subvariety X^u of the varieties of complete quadrics and of complete skew forms that we prove in Section 5 yields a Białynicki-Birula cell decomposition. Also in Section 5 are descriptions of the Poincaré polynomials in each case. Section 6 is devoted to the investigation of the poset of inclusion relations among the closures of cells in Section 6 and the irreducible components in the case of complete quadrics.

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2. **Notation**

We use $\mathbb{N}$ to denote non-negative integers, and if $n \in \mathbb{N}$ is non-zero, then we denote by $[n]$ the set $\{1, 2, \dots, n\}$. All algebraic groups and algebraic varieties are defined over $\mathbb{C}$, though our results extend to those over any algebraically closed field of characteristic 0. We do not assume irreducibility in the definition of an algebraic variety.

Our basic list of notation is as follows:

Mat_n :	space of $n \times n$ matrices
$\mathrm{SL}_n \subset \mathrm{Mat}_n$:	invertible matrices with determinant 1
Sym_n :	space of $n \times n$ symmetric matrices
$\mathrm{Sym}_n^0 \subset \mathrm{Sym}_n$:	invertible symmetric matrices
Skew_n :	space of $n \times n$ skew-symmetric matrices
$\mathrm{Skew}_n^0 \subset \mathrm{Skew}_n$:	invertible skew-symmetric matrices

There is a common left action of SL_n on the spaces $\mathrm{Mat}_n, \mathrm{Sym}_n$ and Skew_n given by

$$g \cdot A = (g^{-1})^{\mathsf{T}} A g^{-1}. \tag{2.1}$$

Note that $\mathrm{Mat}_n \cong \mathrm{Sym}_n \bigoplus \mathrm{Skew}_n$ as $\mathbb{C}[\mathrm{SL}_n]$-modules. It follows from Cholesky's decompositions that on Sym_n^0 and Skew_n^0 the action (2.1) of SL_n is transitive (see Appendix B of [5]). It is easy to check that the stabilizer of the identity matrix $I_n \in \mathrm{Sym}_n^0$ is the special orthogonal group $\mathrm{SO}_n = \{A \in \mathrm{SL}_n : \ AA^\top = I_n\}$, and the stabilizer of

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in \mathrm{Skew}_{2n}^0$$

is the symplectic subgroup $\mathrm{Sp}_{2n} \subset \mathrm{SL}_{2n}$. Therefore, the quotients $\mathrm{SL}_n/\mathrm{SO}_{2n}$ and $\mathrm{SL}_{2n}/\mathrm{Sp}_{2n}$ are canonically identified with the quasi-affine varieties Sym_n^0 and Skew_{2n}^0, respectively.

A *composition* of n is an ordered sequence positive integers that sum to n. For example, $(1,3,2,4,2)$ is a composition of 12. We denote by $Comp(n)$ the set of all compositions of n.

A *partition* of n is an unordered sequence of integers that sum to n. Since it is unordered, without loss of generality we assume that the entries of a partition form a non-increasing sequence. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a partition of n with $\lambda_k \geq 1$. In this case, we call k the length of λ, and denote it by $\ell(\lambda)$. Entries λ_i, $i = 1, \dots, k$ are called the *parts* of λ. Alternatively, we write $\lambda = (1^{\alpha_1}, 2^{\alpha_2}, \dots, l^{\alpha_l})$ to indicate that λ consists of α_1 1's, α_2 2's, and so on. Zero parts or terms with zero exponent may be added and removed without altering λ. For example, each of $(3,3,2)$, $(3,3,2,0)$, $(1^0, 2^1, 3^2)$, and $(2^1, 3^2)$ represent the same partition $3+3+2$ of 8. As a special case used frequently in this paper, the notation (k) indicates the partition with single part equal to k.

Finally, for an arbitrary matrix $B = (b_{ij})_{i,j=1}^n$, we say that b_{ij} lies on the k-th anti-diagonal if $i+j = n+k$. The first anti-diagonal of B is the *main anti-diagonal*.

3. Preliminaries

3.1. Unipotent invariant matrices.

We recall some basic structural results on unipotent fixed matrices from [2].

Let N be a nilpotent matrix with entries in $\mathbb{C}$, and $u = \exp N$. There is an elementary but efficient criterion for determining when a matrix is fixed by u:

$$u \cdot A = A \iff AN + N^{\mathsf{T}} A = 0. \tag{3.1}$$

It is easy to see that the Jordan type of $u = \exp(N)$ is the same as the Jordan type of N. Let N and N' be two nilpotent matrices that are conjugate in SL_n, say $N' = SNS^{-1}$. If we set $u = \exp(N)$ and $u' = \exp(N')$, then the fixed point loci of

u and u' are isomorphic via $A \mapsto SAS^{\mathsf{T}}$. It follows that the fixed point space X^u depends only on the Jordan type of u, or equivalently only on the Jordan type of N.

It is well known that the Jordan classes of $n \times n$ nilpotent matrices are in bijection with partitions of n. Indeed, let $N_{(p)}$ be the p-by-p matrix

$$N_{(p)} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{3.2}$$

Then the above correspondence associates to a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, to the Jordan matrix N_λ given in block form by

$$N_\lambda = \begin{pmatrix} N_{\lambda_1} & 0 & \cdots & 0 \\ 0 & N_{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N_{\lambda_k} \end{pmatrix}.$$

From now on if the Jordan type of u is λ, we write X^λ in place of X^u. If λ is given but u is left unspecified, then it is assumed to be $\exp(N_\lambda)$.

It is straightforward to verify that a generic element $A = (a_{i,j})_{i,j=1}^n$ of $\mathrm{Mat}_n^{(n)}$ has the following form:

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & a_{1,n} \\ 0 & 0 & 0 & \cdots & 0 & -a_{1,n} & a_{2,n} \\ 0 & 0 & 0 & \cdots & a_{1,n} & -a_{2,n} & a_{3,n} \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots \\ 0 & 0 & (-1)^{n-3}a_{1,n} & \cdots & a_{n-4,n} & -a_{n-3,n} & a_{n-2,n} \\ 0 & (-1)^{n-2}a_{1,n} & (-1)^{n-3}a_{2,n} & \cdots & a_{n-3,n} & -a_{n-2,n} & a_{n-1,n} \\ (-1)^{n-1}a_{1,n} & (-1)^{n-2}a_{2,n} & (-1)^{n-3}a_{3,n} & \cdots & a_{n-2,n} & -a_{n-1,n} & a_{n,n} \end{pmatrix} \tag{3.3}$$

An analysis of the matrices of the form (3.3) yields the following propositions.

Proposition 3.4. *Let $N = N_{(n)}$ and let $A = (a_{i,j})_{i,j=1}^n$ be a symmetric matrix satisfying $AN + N^{\mathsf{T}}A = 0$. Then*

1. *If n is even, then A has the form (3.3) with $a_{2l+1,n} = 0$ for all l.*
2. *If n is odd, then A has the form (3.3) with $a_{2l,n} = 0$ for all l.*

Proposition 3.5. *Let $N = N_{(n)}$ and let $A = (a_{i,j})_{i,j=1}^n$ be a skew-symmetric matrix satisfying $AN + N^{\mathsf{T}}A = 0$. Then*

1. *If n is even, then A has the form (3.3) with $a_{2l,n} = 0$ for all l.*
2. *If n is odd, then A has the form (3.3) with $a_{2l+1,n} = 0$ for all l.*

Remark 3.6. Note that it follows from these two Propositions that Sym_n^0 (resp., Skew_n^0) contain an $\exp(N_{(n)})$-fixed quadric if and only if n is odd (resp., even). We are going to use this observation in the proof of Proposition 4.5.

3.2. **One-parameter subgroups.** Let $T \subseteq \mathrm{SL}_n$ be a maximal torus. Define a one parameter subgroup (a *1-PSG*, for short) $\phi_n : \mathbb{C}^* \to T \subset \mathrm{SL}_n$ by

$$\begin{cases} \phi_n(s) := \mathrm{diag}(s^k, \ldots, s, 1, s^{-1}, \ldots, s^{-k}) = \mathrm{diag}(s^{k-i+1})_{i=1}^{2k+1} & \text{if } n = 2k+1; \\ \phi_n(s) := \mathrm{diag}(s^{2k-1}, s^{2k-3}, \ldots, s^1, s^{-1}, \ldots, s^{-(2k-1)}) = \mathrm{diag}(s^{2k-(2i-1)})_{i=1}^{2k} & \text{if } n = 2k. \end{cases}$$

Let $t \in \mathbb{C}$, and define $u_n(t) := \exp(tN_{(n)}) \in \mathrm{SL}_n$, an additive 1-PSG. It is straightforward to calculate

$$u_n(t) = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-2}}{(n-2)!} & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n-3}}{(n-3)!} & \frac{t^{n-2}}{(n-2)!} \\ 0 & 0 & 1 & \cdots & \frac{t^{n-4}}{(n-4)!} & \frac{t^{n-3}}{(n-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & t \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

A straightforward matrix multiplication shows that $\phi_n(s)u_n(t)(\phi_n(s))^{-1}$ is equal to $u_n(st)$ if n is odd and $u_n(s^2t)$ if n is even.

Remark 3.7. Let N be an arbitrary nilpotent matrix with entries in $\mathbb{C}$, $u = \exp N$, and $U = \{\exp(tN) : t \in \mathbb{C}\}$. If X is any complete variety on which U acts, then $X^u = X^U$. (This result fails in characteristic p.)

3.3. **Embeddings.** Let G denote SL_n, $B \subset G$ the subgroup of upper triangular matrices and let $T \subset B$ be the subgroup of diagonal matrices. A closed subgroup P of G is called a *standard parabolic subgroup* if $B \subseteq P$. More generally, a subgroup P' is called *parabolic* if there exists $g \in G$ such that $P' = gPg^{-1}$ for some standard parabolic subgroup $P \subset G$.

The simple roots associated to SL_n comprise $\Delta := \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$. There exists a one-to-one correspondence between the standard parabolic subgroups of G and the subsets $S \subseteq \Delta$ given by

$$S \rightsquigarrow P_S := BW_SB,$$

where W_S is the subgroup of the symmetric group S_n generated by the simple transpositions $s_i = (i, i+1)$ for i such that $\alpha_i \in S$. With this notation, the quotient space G/P_S is called the *partial flag variety* of type S. We need the following interpretation of G/P_S.

Let $J = \{1 \le j \le n-1 : \alpha_j \notin S\} = \{j_1 < \cdots < j_s\}$. Then G/P_S is identified with the set of all nested sequences of vector spaces (flags) of the form

$$\mathcal{F} :\ 0 \subset V_{j_1} \subset V_{j_2} \subset \cdots \subset V_{j_s} \subset \mathbb{C}^n,$$

where $\dim V_{j_k} = j_k$ for $k = 1, \ldots, s$. The *type* of a flag $\mathcal{F}$ is defined to be the set J, or equivalently, it is the composition $(j_1, j_2 - j_1, \ldots, j_s - j_{s-1}, n - j_s)$ of n.

Finally, let us mention the fact that G/P_S has a canonical decomposition into B-orbits where the orbits are indexed by the elements of W^S, the minimal length coset representatives. Furthermore, T acts on G/P_S and each B-orbit contains a unique T-fixed flag [10].

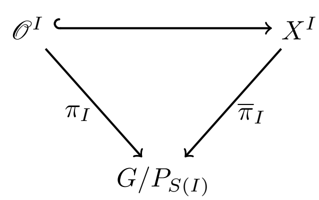
We briefly review the theory of wonderful embeddings, referring the reader to [4] for more details. For us, a symmetric space is a quotient of the form G/K, where G is an algebraic group and K is the normalizer of the fixed subgroup of an automorphism $\sigma : G \to G$ of order 2. When G is a semi-simple simply connected complex algebraic group, there exists a unique minimal smooth projective G-variety $X = X_{G/K}$, called the *wonderful embedding* of G/K, such that

(1) X contains an open G-orbit X_0 isomorphic to G/K;
(2) $X - X_0$ is the union of finitely many G-stable smooth codimension one subvarieties X_i for $i = 1, 2, \ldots, r$;
(3) for any $I \subset [r]$, the intersection $X^I := \bigcap_{i \notin I} X_i$ is smooth and transverse;
(4) every irreducible G-stable subvariety has the form X^I for some $I \subset [r]$;
(5) for $I \subset [r]$, let $\mathscr{O}^I$ denote the dense G-orbit in X^I. There is a fundamental decomposition

$$X^I = \bigsqcup_{K \subset I} \mathscr{O}^K. \tag{3.8}$$

Consequently, G-orbit closures form a Boolean lattice under the inclusion order and there exists a unique closed orbit Z corresponding to $I = \emptyset$.

Wonderful embeddings have a recursive structure. For each $I \subset [r]$, there exists a parabolic subgroup $P_{S(I)}$ and G-equivariant fibrations $\pi : \mathscr{O}^I \to G/P_{S(I)}$ and $\overline{\pi}_I : X^I \to G/P_{S(I)}$ such that the following diagram commutes:



The fiber of π is L/N, where L is the semi-simple part of the Levi subgroup of $P_{S(I)}$ and N is the normalizer in L of the fixed subgroup of θ induced on L. Moreover, the fiber $(\overline{\pi}_I)^{-1}(x)$ over a point $x \in G/P_{S(I)}$ is isomorphic to a wonderful embedding of $\pi_I^{-1}(x)$.

There is more to be said about the parametrizing set $[r]$ of the G-orbits. We choose a torus S, maximal among those consisting of semi-simple elements $x \in G$ such that $\theta(x) = x^{-1}$. Let T be a maximal torus containing S. Then T is automatically θ-stable. Denote by Φ the root system for T in G. Then θ induces an involution on Φ, which we denote by θ, also. Let $\Phi_0 \subseteq \Phi$ denote the set of roots that are fixed by θ, and let Φ_1 denote the complementary set of moved roots. We choose a Borel subgroup B containing T such that the corresponding set of positive roots Φ^+ satisfy $\theta(\Phi^+ \cap \Phi_1) \subseteq -\Phi^+$.

There is a natural restriction map res : $X^*(T) \to X^*(S)$ and the image of Φ_1 gives a not necessarily reduced root system in $X^*(S) \otimes_{\mathbb{Z}} \mathbb{R}$. Recall that Δ denotes the set of simple roots of Φ^+ and set $\Delta_1 = \Delta \cap \Phi_1$. We call the image $\overline{\Delta} = \mathrm{res}(\Delta_1)$, the set of simple restricted roots. Finally, let Π denote the set of fixed simple roots $\Phi_0 \cap \Delta$. We denote by P_Π the corresponding parabolic subgroup of G. The unique closed orbit of X is isomorphic to G/P_Π.

3.4. **Complete Quadrics.** There is a vast literature on the variety of complete quadrics. See [11] for a survey. We briefly recall the necessary definitions.

Two non-degenerate $n \times n$ symmetric matrices represent the same non-degenerate quadric hypersurface in $\mathbb{P}^{n-1}$ if and only if they differ by multiplication by a scalar; hence $\mathbb{P}(\mathrm{Sym}_n)$ is the moduli of such quadric hypersurfaces. In this context, the classical definition of the variety of complete quadrics in $\mathbb{P}^{n-1}$ (see [14, 15, 24]) is as the closure of the image of the map

(3.9)
$$[A] \mapsto ([\Lambda^1(A)], [\Lambda^2(A)], \dots, [\Lambda^{n-1}(A)]) \in \prod_{i=1}^{n-1} \mathbb{P}(\Lambda^i(\mathrm{Sym}_n)),\ [A] \in \mathbb{P}(\mathrm{Sym}_n^0).$$

In more modern group-theoretical language, the variety of complete quadrics is the wonderful embedding $\mathcal{M}_{\mathrm{Q}} = \mathcal{M}_{\mathrm{SL}_n/\mathrm{SO}_n}$ of the symmetric space $\mathrm{SL}_n/\mathrm{SO}_n$.

The rank r of $\mathcal{M}_{\mathrm{Q}}$ is equal to $n-1$ and the parameterizing set $[r] = [n-1]$ for the G-invariant divisors on X is identified with Δ by identifying i and α_i. Let $I \subseteq [n-1]$ and let P_I be the corresponding standard parabolic subgroup. It follows from results of Vainsencher [25] that a point $\mathcal{P} \in \mathcal{M}_{\mathrm{Q}}$ is described by the data of a flag

$$\mathcal{F} : V_0 = 0 \subset V_1 \subset \cdots \subset V_{s-1} \subset V_s = \mathbb{C}^n \tag{3.10}$$

of type $I^c = \{1 \le j \le n-1 : j \notin I\} = \{j_1 < \cdots < j_{s-1}\}$, where $\dim V_k = j_k$, and a collection $\mathcal{Q} = (Q_1, \dots, Q_s)$ of non-singular quadric hypersurces $Q_i \subseteq \mathbb{P}(V_i/V_{i-1})$. Moreover, $\mathcal{O}^I$ consists of complete quadrics whose flag $\mathcal{F}$ is of type I and the SL_n-equivariant map

$$\overline{\pi}_I : \mathcal{M}_{\mathrm{Q}}^I \to \mathrm{SL}_n/P_I. \tag{3.11}$$

restricts to $(\mathcal{F}, \mathcal{Q}) \mapsto \mathcal{F}$ over $\mathcal{O}^I$. The fiber of $\overline{\pi}_I$ over $\mathcal{F} \in \mathrm{SL}_n/P_I$ is isomorphic to a product of varieties of complete quadrics of smaller dimension.

Let $\theta_o : \mathrm{SL}_n \to \mathrm{SL}_n$ denote the automorphism $\theta_o(g) = (g^{-1})^\top$. Then θ_o is an involution and its fixed subgroup is precisely SO_n. In this case, the set of moved roots and the set of simple roots Δ of T coincide. Here T is the maximal torus of diagonal matrices in SL_n. The unique closed orbit, therefore, corresponds to the empty set of roots, and hence, it is isomorphic to the flag variety G/B.

Furthermore, there is a one-to-one correspondence between the subsets of $\Delta_1 = \Delta$ and the G-orbits. It follows for each standard parabolic subgroup $P_I \subseteq G$, $I \subset \Delta$, there exists G-equivariant fibration $\pi : \mathscr{O}^I \to G/P_I$ with fibers isomorphic to L_I/N_I, where L_I is the semi-simple part of the Levi subgroup of P_I and N_I is the normalizer in L_I of the fixed subgroup of θ induced on L_I.

For any subgroup $H \subseteq GL_n$, let $S(H)$ denote the subgroup of elements of H of determinant 1 and let $\tilde{H}$ denote the subgroup of GL_n generated by H and the scalar multiples of the identity matrix. The standard Levi subgroups in $G = \mathrm{SL}_n$ are of the form $S(\prod_{j=1}^k \mathrm{GL}_{m_j})$ with $\sum m_j = n$. Since the θ-fixed subgroup of such a group is isomorphic to $S(\prod_{j=1}^k \widetilde{\mathrm{SO}}_{m_j})$, a fiber of π has to form $\prod_{j=1}^k \mathrm{Sym}_{m_j}^0$.

3.5. **Complete Skew Forms.** From now on, n is assumed to be an even number whenever we deal with a symplectic group.

The classical construction of the wonderful compactification of $\mathrm{SL}_n/\mathrm{Sp}_n$, $\mathcal{M}_{\mathrm{S}} := \mathcal{M}_{\mathrm{SL}_n/\mathrm{Sp}_n}$, is similar to that of $\mathcal{M}_{\mathrm{Q}}$; instead of symmetric matrices in (3.9) one needs to use $n \times n$ skew-symmetric matrices. See [23] for details.

The group-theoretical definition parallels that of $\mathcal{M}_{\mathrm{Q}}$ as well. A point $\mathcal{P} \in \mathcal{M}_{\mathrm{S}}$ is given by the data of a flag of even dimensional subspaces

$$\mathcal{F} : V_0 = 0 \subset V_1 \subset \cdots \subset V_{s-1} \subset V_s = \mathbb{C}^n \tag{3.12}$$

and a collection $\mathcal{W} = (\omega_1, \dots \omega_s)$ of 2-forms, where ω_i is a non-degenerate 2-form in $\mathbb{P}(V_i/V_{i-1})$. Moreover, $\mathscr{O}^I$ consists of complete 2-forms whose flag $\mathcal{F}$ is of type I and the SL_n-equivariant map

$$\overline{\pi}_I : \mathcal{M}_{\mathrm{S}}^I \to \mathrm{SL}_n/P_I. \tag{3.13}$$

restricts to $(\mathcal{F}, \mathcal{W}) \mapsto \mathcal{F}$ over $\mathscr{O}^I$. The fiber of $\overline{\pi}_I$ over $\mathcal{F} \in \mathrm{SL}_n/P_I$ is isomorphic to a product of wonderful compactifications of spaces of complete skew forms of smaller dimension. The rank r is equal to $n/2 - 1$ and the subsets of $[r]$ can be identified with subsets of even numbers in $[n-1]$, which gives the natural identification with certain subsets of Δ.

4. **Fixed Subvarieties**

In this section we describe the fixed subvarieties of a regular unipotent element operating on the varieties of complete quadrics and complete skew forms.

Definition 4.1. Let

$$\mathcal{F} : 0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k = V$$

be a (partial) flag of $V = \mathbb{C}^n$. For any $g \in \mathrm{SL}_n$, the flag $\mathcal{F}$ is g-fixed if and only if $g(V_i) = V_i$ for all $1 \le i \le k$. For any $n \times n$ matrix M, the flag $\mathcal{F}$ is M-stable if and only if $M(V_i) \subseteq V_{i-1}$ for all $1 \le i \le k$.

Remark 4.2. Let N be a nilpotent $n \times n$ matrix and $u = \exp(N)$ the corresponding unipotent element of SL_n. Then a flag $\mathcal{F}$ of $\mathbb{C}^n$ is u-fixed if and only if it is N-stable.

Lemma 4.3. *Let $V = \mathbb{C}^n$ with standard basis $\{e_1, e_2, \dots, e_n\}$, and let*

$$N = N_{(n)} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

be the regular nilpotent operator for this basis. Set $u = \exp(N)$. Then for each composition γ of n, there is a unique flag of type γ that is fixed by u, namely the sub-flag of type γ of the standard complete flag $0 \subset \mathbb{C}e_1 \subset \mathbb{C}e_1 \oplus \mathbb{C}e_2 \subset \cdots \subset \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \cdots \oplus \mathbb{C}e_n = V$.

Proof. We prove this by induction on the number k of parts of the composition γ. Let v be any non-zero vector in V_1 and let j be the largest integer such that v contains a non-zero e_j component. Then $v, N(v), N^2(v), \dots, N^{j-1}(v)$ are j linearly independent vectors in V_1. Since $\dim(V_1) = \gamma_1$, this implies $j \le \gamma_1$. Thus $V_1 = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \cdots \oplus \mathbb{C}e_{\gamma_1}$. Now apply the inductive hypothesis to the induced flag obtained by quotienting $\mathcal{F}$ by V_1, noting that the induced nilpotent transformation on V/V_1 has Jordan type $(n - \gamma_1)$, allowing the induction to proceed. □

Remark 4.4. Let $u = \exp(N)$ be as in Lemma 4.3, and let $\mathcal{F} : 0 \subset V_1 \subset V_2 \subset \cdots \subset V_k = \mathbb{C}^n$ be the N-stable (or equivalently, u-fixed) flag of type $\gamma = (\gamma_1, \dots, \gamma_k)$. Then restriction of N to V_i/V_{i-1} is another principal nilpotent operator and its Jordan type is (γ_i). By Remark 3.6, if $\dim(V_i/V_{i-1})$ is even for some $1 \le i \le k$, then there are no u-fixed non-singular quadrics in $\mathbb{P}(V_i/V_{i-1})$. Consequently, there cannot be any u-fixed complete quadrics on such a flag $\mathcal{F}$. On the other hand, if $\dim(V_i/V_{i-1})$ is odd for some $1 \le i \le k$, then there are no u-fixed non-degenerate 2-forms in $\mathbb{P}(V_i/V_{i-1})$, hence no u-fixed complete skew-forms on such a flag.

We now give a set theoretic description of the u-fixed subvarieties of the two compactifications.

Proposition 4.5. *The unipotent fixed subvariety $\mathcal{M}_Q^{(n)}$ has a decomposition into affine spaces indexed by the compositions of n with no even parts. Similarly, the unipotent fixed subvariety $\mathcal{M}_S^{(n)}$ has a decomposition into affine spaces indexed by the compositions of n with no odd parts.*

Proof. We begin with the case $\mathcal{M}_{\mathrm{Q}}^{(n)}$. Let $(\mathcal{F}, \mathcal{Q})$ be a point of $\mathcal{M}_{\mathrm{Q}}^{(n)}$ with $\mathcal{F}$ of type $\gamma = (\gamma_1, \dots, \gamma_k)$. Then by Remark 4.4 we know that each γ_i $(1 \leq i \leq k)$ is odd. It follows from Proposition 3.4 (1) that the space of non-degenerate u-fixed quadrics on $\mathbb{P}(V_i/V_{i-1})$ is an affine space of dimension $(\gamma_i - 1)/2$.

Let $\mathcal{M}_{\mathrm{Q}}^{(n),\gamma} \subseteq \mathcal{M}_{\mathrm{Q}}^{(n)}$ denote the subspace consisting of complete quadrics of flag type γ. Clearly,

$$\mathcal{M}_{\mathrm{Q}}^{(n)} = \bigsqcup_{\gamma} \mathcal{M}_{\mathrm{Q}}^{(n),\gamma},$$

where the union is over all compositions $\gamma = (\gamma_1, \dots, \gamma_k)$ of n with odd parts only. Moreover, the above discussion implies that $\mathcal{M}_{\mathrm{Q}}^{(n),\gamma}$ is an affine space:

$$\mathcal{M}_{\mathrm{Q}}^{(n),\gamma} = \mathbb{A}^{(\gamma_1-1)/2} \times \cdots \times \mathbb{A}^{(\gamma_k-1)/2} \cong \mathbb{A}^{(n-k)/2}.$$

The argument for $\mathcal{M}_{\mathrm{S}}^{(n)}$ is similar, so we write the end result:

$$\mathcal{M}_{\mathrm{S}}^{(n)} = \bigsqcup_{\beta} \mathcal{M}_{\mathrm{S}}^{(n),\beta},$$

where the union is over all compositions $\beta = (\beta_1, \dots, \beta_k)$ of n with even parts only, and each $\mathcal{M}_{\mathrm{S}}^{(n),\beta}$ is an affine space:

$$\mathcal{M}_{\mathrm{S}}^{(n),\beta} = \mathbb{A}^{\beta_1/2-1} \times \cdots \times \mathbb{A}^{\beta_k/2-1} \cong \mathbb{A}^{n/2-k}.$$

□

As an immediate corollary of the proof of Proposition 4.5 we have

Corollary 4.6. *The dimensions of the two u-fixed loci of the respective compactifications are given by $\dim \mathcal{M}_Q^{(n)} = \lfloor (n-1)/2 \rfloor$, and $\dim \mathcal{M}_S^{(n)} = n/2 - 1$.*

5. **Cell Decompositions**

5.1. **$\mathbb{C}^*$ action on matrices.** Let $T \subset \mathrm{SL}_n$ denote the maximal torus of diagonal matrices. We compute the stabilizer of T acting on $X^{(n)}$, where X is any of the spaces $X = \mathrm{Mat}_n, X = \mathrm{Skew}_n$, or $X = \mathrm{Sym}_n$.

Let Y denote the set of invertible elements from X. By (3.3) and Propositions 3.4 and 3.5, the set $Y^{(n)}$ is non-empty if and only if the main anti-diagonal of a generic element from Y has no zero entries. If $Y = \mathrm{GL}_n$, this is independent of n; however, by Propositions 3.4 and 3.5, if $Y = \mathrm{Sym}_n^0$, then n has to be odd, and if $Y = \mathrm{Skew}_n^0$, then n must be even.

Let $t \in T$ be such that $t^{-1} = \mathrm{diag}(t_1, \dots, t_n)$, $t_i \in \mathbb{C}^*$, and let $A = (a_{i,j}) \in Y^{(n)}$ be a generic element. It is straightforward to verify that

$$t \cdot A = (t_i t_j a_{i,j})_{i,j=1}^n \in Y^{(n)} \iff t_i t_j = t_k t_l \text{ for all } i + j = k + l. \tag{5.1}$$

Example 5.2. Let A be a generic element of $X^{(6)} = \mathrm{Mat}_6^{(6)}$. Then

$$t{\cdot}A = (t^{-1})^\top A(t^{-1}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & t_1t_6a_{11} \\ 0 & 0 & 0 & 0 & -t_2t_5a_{11} & t_2t_6a_{12} \\ 0 & 0 & 0 & t_3t_4a_{11} & -t_3t_5a_{12} & t_3t_6a_{13} \\ 0 & 0 & -t_4t_3a_{11} & t_4^2a_{12} & -t_4t_5a_{13} & t_4t_6a_{14} \\ 0 & t_5t_2a_{11} & -t_5t_3a_{12} & t_5t_4a_{13} & -t_5^2a_{14} & t_5t_6a_{15} \\ -t_6t_1a_{11} & t_6t_2a_{12} & -t_6t_3a_{13} & t_6t_4a_{14} & -t_6t_5a_{15} & t_6^2a_{16} \end{pmatrix}.$$

Let $\phi_n : \mathbb{C}^* \to T \subset \mathrm{SL}_n$ denote the 1-PSG defined in Section 3.2. Observe that if $a_{i,j}$ is the (i,j)-th entry of $A \in X$, then the (i,j)-th entry of $\phi_n(s)\cdot A$ is $s^{i+j-(n+1)}a_{i,j}$. Therefore, both $X^{(n)}$ and $Y^{(n)}$ are stable under $\mathbb{C}^*$ action defined by $\phi_n(s)$. Moreover, $A \in Y^{(n)}$ is a fixed point of this $\mathbb{C}^*$-action if and only if the only non-zero entries of A appear along its main anti-diagonal.

Example 5.3. Let $\phi_6(s) = \mathrm{diag}(s^5, s^3, s^1, s^{-1}, s^{-3}, s^{-5})$, and let $A \in X^{(6)}$, $X = \mathrm{Skew}_6$ denote a generic element, which is of the form

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & a_{11} \\ 0 & 0 & 0 & 0 & -a_{11} & 0 \\ 0 & 0 & 0 & a_{11} & 0 & a_{13} \\ 0 & 0 & -a_{11} & 0 & -a_{13} & 0 \\ 0 & a_{11} & 0 & a_{13} & 0 & a_{15} \\ -a_{11} & 0 & -a_{13} & 0 & -a_{15} & 0 \end{pmatrix}.$$

Then

$$s\cdot A = \phi(s^{-1})^\top A\phi(s^{-1}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & a_{11} \\ 0 & 0 & 0 & 0 & -a_{11} & 0 \\ 0 & 0 & 0 & a_{11} & 0 & s^4a_{13} \\ 0 & 0 & -a_{11} & 0 & -s^4a_{13} & 0 \\ 0 & a_{11} & 0 & s^4a_{13} & 0 & s^8a_{15} \\ -a_{11} & 0 & -s^4a_{13} & 0 & -s^8a_{15} & 0 \end{pmatrix}.$$

Lemma 5.4. *Let $\phi_n : \mathbb{C}^* \to T$ be defined as in Section 3.2. Let Y be either Skew_n^0 or Sym_n^0. Then $\mathbb{C}^*$ acts on $\mathbb{P}(Y^{(n)})$ via ϕ_n with a unique fixed point. Moreover, the unique $\mathbb{C}^*$-fixed point is*

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ -1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

5.2. **Carrell-Goresky's Białynicki-Birula cell decomposition.** Under a "good" $\mathbb{C}^*$-action, a variety decomposes into affine subspaces, and the classes of the closures of these affine subspaces form a basis for the Chow ring. We briefly recall certain aspects of this well developed theory of torus actions.

Theorem 5.5. [1, Theorem 4.3] *Let Z be a smooth, irreducible projective variety with a $\mathbb{C}^*$-action such that the fixed point set is finite. Then*

(1) *For each fixed point $x \in Z^{\mathbb{C}^*}$, the set*

$$C_x := \{y \in Z : \lim_{t\to 0} t\cdot y = x\}$$

is an affine space.

(2) *The closures $\overline{C_x}$ of the cells C_x form an additive basis for the Chow ring (as well as the integral cohomology ring) of Z.*

It turns out that the smoothness condition in the above theorem can be relaxed.

Theorem 5.6. [3, Theorem 1] *Let Z be a possibly singular, possibly reducible $\mathbb{C}^*$-variety whose fixed point set is finite. If the first statement in Theorem 5.5 holds, then so does the second statement.*

If the first statement of Theorem 5.5 holds (hence the conclusion of Theorem 5.6 holds), then we call the decomposition $Z = \bigsqcup_{x \in Z^{\mathbb{C}^*}} C_x$ *a cell decomposition* of Z with respect to the $\mathbb{C}^*$-action, and refer to the affine sets C_x, $x \in Z^{\mathbb{C}^*}$, as the *cells* of the decomposition.

5.3. **Torus action on compactifications.** We consider extensions of the action given by ϕ_n (defined in Section 3.2) on matrices to $\mathcal{M}_{\mathrm{Q}}$ and to $\mathcal{M}_{\mathrm{S}}$. Recall from the proof of Proposition 4.5 that there are decompositions

$$\mathcal{M}_{\mathrm{Q}}^{(n)} = \bigsqcup_{\gamma} \mathcal{M}_{\mathrm{Q}}^{(n),\gamma} \quad \text{and} \quad \mathcal{M}_{\mathrm{S}}^{(n)} = \bigsqcup_{\beta} \mathcal{M}_{\mathrm{S}}^{(n),\beta}, \tag{5.7}$$

where $\mathcal{M}_{\mathrm{Q}}^{(n),\gamma}$ is the affine space consisting of unipotent invariant complete quadrics of flag type γ, and $\mathcal{M}_{\mathrm{S}}^{(n),\beta}$ is the affine space consisting of unipotent invariant complete skew-forms of flag type β.

Theorem 5.8. *Let ϕ_n be the 1-PSG defined in Section 3.2. Then $\mathbb{C}^*$ acts on the spaces $\mathcal{M}_Q^{(n)}$ and $\mathcal{M}_S^{(n)}$ with finitely many fixed points. The cells of the torus action are given by the decompositions (5.7).*

Proof. Let $e_1, \dots, e_n$ denote the standard ordered basis for $\mathbb{C}^n$, and let $\mathcal{F}_s$ denote the standard flag $0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n$, where V_r is the $\mathbb{C}$-span of $e_1, \dots, e_r$. Let $x = (\mathcal{F}, \mathcal{P})$ be either from $\mathcal{M}_{\mathrm{Q}}^{(n)}$ or from $\mathcal{M}_{\mathrm{S}}^{(n)}$. By Lemma 4.3, we know that in order for x to be fixed by $u = \exp N_{(n)}$ its flag $\mathcal{F}$ has to be a subflag of $\mathcal{F}_s$. Suppose $\mathcal{F}:\ 0 \subset V_{j_1} \subset \cdots \subset V_{j_k} = \mathbb{C}^n$, and suppose $\mathcal{P} = (P_1, \dots, P_k)$ is the (symmetric or skew) form of x. Then the i-th entry of $\mathcal{P}$ is a non-degenerate form (up to a scaler multiple) on the quotient space V_{j_i}/V_{j_i-1}, and furthermore, it is fixed by the restriction of u on V_{j_i}/V_{j_i-1}, which is a regular unipotent operator of Jordan type $(\dim V_{j_i} - \dim V_{j_{i-1}})$.

It is straightforward to verify that $\phi_n(\mathbb{C}^*) \cdot \mathcal{F} = \mathcal{F}$. Therefore, it remains showing that there are only finitely many $\mathcal{P}$ on $\mathcal{F}$ such that $\phi(\mathbb{C}^*) \cdot \mathcal{P} = \mathcal{P}$. Note that this equality is true if and only if the restriction of $\phi_n(\mathbb{C}^*)$ to each quotient space $V_{j_i}/V_{j_{i-1}}$ acts on the corresponding form P_i trivially. But we know from Lemma 5.4 that P_i has to be represented by an anti-diagonal matrix of size $\dim V_{j_i} - \dim V_{j_{i-1}}$. In either case, our conclusion is that there is only one $\mathbb{C}^*$-fixed point for each subflag $\mathcal{F}$ of the standard flag $\mathcal{F}_s$. Since $\mathbb{C}^*$-action preserves the flag type, and since the affine subsets $\mathcal{M}_{\mathrm{Q}}^{(n),\gamma}$, $\mathcal{M}_{\mathrm{S}}^{(n),\beta}$ from (5.7) are precisely the unipotent invariant complete forms of a given flag type, by Theorem 5.6 we see that the affine subsets of (5.7) must be the cells of the $\mathbb{C}^*$-action. □

Let $Comp_o(n)$ denote the set of all compositions of n with odd parts only, and let $Comp_e(n)$ denote the set of all compositions of n with even parts only.

Corollary 5.9. *The Poincaré polynomial $P_{\mathcal{M}_Q^{(n)}}(x)$ of $\mathcal{M}_Q^{(n)}$ is given by*

$$P_{\mathcal{M}_Q^{(n)}}(x) = \sum_{(\gamma_1,\dots,\gamma_k)\in Comp_o(n)} x^{(n-k)/2} = \sum_{i=0}^{n} \binom{n-1-i}{i} x^i.$$

Proof. For simplicity let $p_n(x)$ denote the Poincaré polynomial $P_{\mathcal{M}_{\mathrm{Q}}^{(n)}}(x)$. The first equality follows from the proof of Proposition 4.5. To prove the second equality, we first find a recurrence for $p_n(x)$. To this end, observe that $Comp_o(n) = Comp_o(n)' \sqcup Comp_o(n)''$, where $Comp_o(n)'$ consists of elements $\gamma \in Comp_o(n)$ such that $\gamma_1 = 1$, and $Comp_o(n)''$ consists of those $\rho \in Comp_o(n)$ with $\rho_1 > 1$.

$$\begin{aligned} p_n(x) &= \sum_{\gamma=(\gamma_1,\dots,\gamma_k)\in Comp_o(n)'} x^{(n-k)/2} + \sum_{\rho=(\rho_1,\dots,\rho_l)\in Comp_o(n)''} x^{(n-l)/2} \\ &= \sum_{(\gamma_2,\dots,\gamma_k)\in Comp_o(n-1)} x^{[(n-1)-(k-1)]/2} + \sum_{(\rho_1-2,\dots,\rho_l)\in Comp_o(n-2)} x \cdot x^{[(n-2)-l]/2}, \end{aligned}$$

which simplifies to

$$p_n(x) = p_{n-1}(x) + x p_{n-2}(x). \tag{5.10}$$

It is easy to easy to check that $p_1(x) = p_2(x) = 1$. Now, a simple induction argument combined with the well known recurrence

$$\binom{n-1-i}{i} = \binom{n-2-i}{i-1} + \binom{n-2-i}{i}$$

together with (5.10) give us the desired second equality. □

Remark 5.11. It follows from (5.10) and the initial conditions that the value at $x = 1$ of the Poincaré polynomial $P_{\mathcal{M}_{\mathrm{Q}}^{(n)}}(x)$ is the $(n-1)$-st Fibonacci number.

Corollary 5.12. *Suppose $n = 2m$. Then the Poincaré polynomial $P_{\mathcal{M}_S^{(n)}}(x)$ of $\mathcal{M}_S^{(n)}$ is given by*

$$P_{\mathcal{M}_S^{(n)}}(x) = \sum_{(\beta_1,\dots,\beta_k)\in Comp_e(n)} x^{(n/2)-k} = (1+x)^{m-1}.$$

Proof. The first equality follows from the proof of Proposition 4.5. To prove the second equality observe that $Comp_e(n)$ is in bijection with $Comp(m)$, the set of all compositions of m. Thus, the equality

$$P_{\mathcal{M}_{\mathrm{S}}^{(n)}}(x) = \sum_{(\beta_1,\dots,\beta_k)\in Comp(m)} x^{m-k} \tag{5.13}$$

is obvious.

Recall the bijection between the set of all compositions of m and the set of all subsets of $[m-1]$: For $\gamma = (\gamma_1,\dots,\gamma_k)$ a composition of m, let I_γ denote the complement of the set $I = \{\gamma_1, \gamma_1+\gamma_2, \dots, \gamma_1+\dots+\gamma_{k-1}\}$ in $[m-1]$. Then the assignment $\gamma \mapsto I_\gamma$ is the desired bijection between compositions of m with k parts and the subsets of $[m-1]$ with $m-1-(k-1) = m-k$ elements. Therefore, the right hand side of equation (5.13) is equal to $\sum_{I\subseteq[m-1]} x^{|I|}$, which is obviously equal to $(1+x)^{m-1}$. □

6. The poset of cell closures

The purpose of this section is to workout the poset structure on the closures of cells of the unipotent fixed subvariety $\mathcal{M}_{\mathrm{Q}}^{(n)}$. As a consequence we observe that the unipotent fixed subvariety $\mathcal{M}_{\mathrm{Q}}^{(n)}$ does not need to be equidimensional.

Given a Białnicki-Birula cell decomposition of a smooth projective variety X, it is not always true that the closure of a cell is the union of other cells. In other words, a Białnicki-Birula decomposition is not always a stratification. However, it is still a very interesting question to study the partial order on the cells C_x, $x \in X^T$ defined by

$$C_x \leq C_y \iff C_x \subseteq \overline{C_y}, \quad \text{for } x, y \in X^T.$$

This motivates the following definition.

Definition 6.1. Let $\rho = (\rho_1, \ldots, \rho_l)$ and $\gamma = (\gamma_k, \ldots, \gamma_k)$ be two compositions from $Comp_o(n)$. Define

$$\rho \leq \gamma \iff \mathcal{M}_Q^{(n),\rho} \subseteq \overline{\mathcal{M}_Q^{(n),\gamma}}.$$

We call $(Comp_o(n), \leq)$ the nth Fibonacci poset.

Proposition 6.2. *Let $\gamma = (\gamma_1, \ldots, \gamma_k), \rho = (\rho_1, \ldots, \rho_l) \in Comp_o(n)$. Then γ covers ρ if and only if there exists a part γ_i of γ which is > 1 and ρ is obtained from γ by replacing γ_i by $\gamma_i - 2, 1, 1$. In other words, $\rho = (\gamma_1, \ldots, \gamma_{i-1}, \gamma_i - 2, 1, 1, \gamma_{i+1}, \ldots, \gamma_k)$. Moreover,*

$$\overline{\mathcal{M}_Q^{(n),\gamma}} = \mathbb{P}^{(\gamma_1 - 1)/2} \times \cdots \times \mathbb{P}^{(\gamma_k - 1)/2}.$$

Proof. This follows from the following explicit description of the the cell $\mathcal{M}_Q^{(n),\gamma}$:

$$\mathcal{M}_Q^{(n),\gamma} \cong \prod_{i=1}^{k} \mathcal{A}_{\gamma_i},$$

where

$$\mathcal{A}_m = \left\{ \begin{bmatrix} 0 & \cdots & 0 & 0 & a_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & a_{r-2} & 0 & a_{r-1} \\ 0 & \cdots & 0 & -a_{r-1} & 0 \\ a_1 & \cdots & a_{r-1} & 0 & a_r \end{bmatrix} : a_i \in \mathbb{C}, a_1 \neq 0 \right\}$$

for $m = 2r + 1$ odd. Note that $\mathcal{A}_m \cong \mathbb{A}^{(m-1)/2}$. It is easy to see that $\overline{\mathcal{A}_m} = \mathcal{A}_m \cup \mathcal{A}_{m-2} \cup \cdots \cup \mathcal{A}_1$, the standard decomposition of $\mathbb{P}^{(m-1)/2}$. □

Corollary 6.3. *Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k) \in Comp_o(n)$ and let $\hat{0} = (1, 1, \ldots, 1)$, the minimal element of $Comp_o(n)$. The interval $[\hat{0}, \gamma]$ is a product of chains. More specifically,*

$$[\hat{0}, \gamma] = C_{(\gamma_1 - 1)/2} \times C_{(\gamma_2 - 1)/2} \times \cdots \times C_{(\gamma_k - 1)/2},$$

where C_m denotes the chain with $m + 1$ elements.

Next, we prove that our ordering is ranked.

Lemma 6.4. *The nth Fibonacci poset is a graded poset. For $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k) \in Comp_o(n)$, the rank function is given by*

$$\ell(\gamma) = (n - k)/2 = \dim \mathcal{M}_Q^{(n),\gamma}.$$

Proof. Clearly, if $\rho \leq \gamma$, then $\ell(\rho) = \dim \mathcal{M}_Q^{(n),\rho} \leq \dim \mathcal{M}_Q^{(n),\gamma} = \ell(\gamma)$. Also, if γ covers ρ, then by Proposition 6.2, ρ is obtained from γ by replacing some γ_i by $\gamma_i - 2, 1, 1$. Thus

$$\ell(\rho) = (n - (k+2))/2 = \ell(\gamma) - 1.$$

□

Note that irreducible components of $\mathcal{M}_Q^{(n),\gamma}$ correspond to maximal elements of the poset $(Comp_o(n), \leq)$. It turns out that $Comp_o(n)$ has more than one maximal element when $n \geq 4$.

Example 6.5. The Hasse diagram for $Comp_o(6)$ is shown in Figure 6.1.

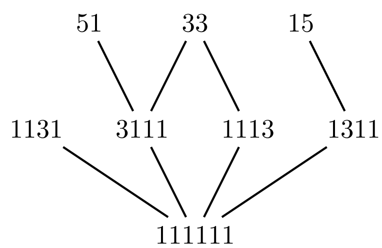


FIGURE 6.1. Hasse diagram for $\mathcal{M}_Q^{(6)}$.

Remark 6.6. We see from Figure 6.1 that $\mathcal{M}_Q^{(6)}$ has four irreducible components: two of which are isomorphic to $\mathbb{P}^2$, one isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and one isomorphic to $\mathbb{P}^1$. Thus, unlike the case of Springer fibers, the irreducible components of $\mathcal{M}_Q^{(n)}$ are not equidimensional.

Our next goal is to study lattice properties of $(Comp_o(n), \leq)$. We start with a preliminary observation.

Lemma 6.7. *Let $\gamma = (\gamma_1, \dots, \gamma_k)$ and $\tau = (\tau_1, \dots, \tau_l)$ be two elements from $Comp_o(n)$ such that $\gamma_1 > 1$ and $\tau_1 = 1$. If $\tau \leq \gamma$, then $\tau_1 = \cdots = \tau_{\gamma_1} = 1$.*

Proof. Since $\gamma \geq \tau$, there exists a sequence of coverings $\gamma = \gamma^{(0)} \geq \gamma^{(1)} \geq \cdots \geq \gamma^{(s)} = \tau$ and a subsequence $\gamma = \gamma^{(i_0)} \geq \gamma^{(i_1)} \geq \cdots \geq \gamma^{(i_d)} = \tau$ such that

$$\gamma_1^{(i_1)} = \gamma_1 - 2 \text{ and } \gamma_2^{(i_1)} = \gamma_3^{(i_1)} = 1,$$
$$\gamma_1^{(i_2)} = \gamma_1^{(i_1)} - 2 \text{ and } \gamma_2^{(i_2)} = \gamma_3^{(i_2)} = \gamma_4^{(i_2)} = \gamma_5^{(i_2)} = 1,$$
$$\vdots$$
$$\gamma_1^{(i_d)} = \gamma_1^{(i_{d-1})} - 2 = 1 \text{ and } \gamma_2^{(i_d)} = \cdots = \gamma_{2d+1}^{(i_d)} = 1.$$

It follows that $\tau_1 = \cdots = \tau_j = 1$ for some $j \geq 2d + 1 = \gamma_1$. □

Recall the partition of $Comp_o(n)$

$$Comp_o(n) = Comp_o(n)' \sqcup Comp_o(n)'',$$

based on the last part given in the proof of Corollary 5.9. There are bijections $\phi : Comp_o(n)' \to Comp_o(n-1)$,

$$\phi(\gamma) = (\gamma_2, \gamma_3, \dots, \gamma_k) \tag{6.8}$$

and $\psi : Comp_o(n)'' \to Comp_o(n-2)$,

$$\psi(\gamma) = (\gamma_1 - 2, \gamma_2, \gamma_3, \dots, \gamma_k). \tag{6.9}$$

Theorem 6.10. *The Fibonacci poset $(Comp_o(n), \leq)$ is a meet-semi lattice.*

Proof. We induct on n. Assume that the statement is true for $Comp_o(m)$ with $m < n$. If γ and ρ both lie in $Comp_o(n)'$ or both lie in $Comp_o(n)''$, then by the induction hypothesis there exists a unique $\gamma \wedge \rho$. Therefore, without loss of generality we assume that $\gamma \in Comp_o(n)''$ and $\rho \in Comp_o(n)'$.

Let $\widetilde{\gamma} \in Comp_o(n)$ be the element obtained from γ by replacing γ_1 by the sequence $1, \dots, 1$ of γ_1 1's. Then $\widetilde{\gamma} \in Comp_o(n)'$, and by the induction hypothesis $\widetilde{\gamma} \wedge \rho \in Comp_o(n)'$ exists uniquely. If $\tau \leq \gamma$ and $\tau \leq \rho$, then by Lemma 6.7, $\tau \leq \widetilde{\gamma}$, so $\tau \leq \widetilde{\gamma} \wedge \rho$. □

In [22], Steinberg shows that the number of irreducible components of the Springer fiber of Jordan type λ equals the number of standard Young tableaux of shape λ. The number of irreducible components of $\mathcal{M}_{\mathrm{Q}}^{(n)}$ satisfies a simple linear recurrence.

Theorem 6.11. *Let a_n be the number of irreducible components of $\mathcal{M}_Q^{(n)}$. Then*

$$a_n = a_{n-1} + a_{n-3} \qquad \text{for } n \geq 4.$$

The initial values of the sequence are $a_1 = a_2 = a_3 = 1$.

Proof. Let $Comp_o(n)^{\max} \subseteq Comp_o(n)$ denote the set of maximal elements in $Comp_o(n)$. From Proposition 6.2, a composition $\gamma \in Comp_o(n)$ belongs to $Comp_o(n)^{\max}$ if and only if the composition contains no consecutive 1's, except possibly two consecutive 1's at the beginning.

Let A, respectively A', be the subsets of $Comp_o(n)^{\max}$ consisting of elements whose last part is 1, respectively ≥ 5. Let $B \subset Comp_o(n)^{\max}$ be the subset consisting of elements whose last part is 3. $Comp_o(n)$ is a disjoint union of A, A', and B.

Define $f : A \cup A' \to Comp_o(n-1)^{\max}$ by sending $\gamma = (\gamma_1, \dots, \gamma_k)$ to $(\gamma_1, \dots, \gamma_{k-1})$ if γ is in A, and to $(\gamma_1, \dots, \gamma_k - 2, 1)$ if γ is in A'. Similarly, define $g : B \to Comp_o(n-3)^{max}$ by $g(\gamma) = (\gamma_1, \dots, \gamma_{k-1})$. The maps f and g are bijections, and therefore,

$$a_n = |Comp_o(n)^{\max}| = |Comp_o(n-1)^{\max}| + |Comp_o(n-3)^{\max}| = a_{n-1} + a_{n-3}.$$

□

Corollary 6.12. *The (ordinary) generating function for a_n, $n \geq 0$ (with the convention $a_0 = 1$) is*

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{1 - x - x^3}.$$

This sequence also counts the number of compositions of $n-2$ into parts 1 and 2 with no consecutive 2's.

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