Spherical product of a partial flag variety with a symmetric space

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Abstract

We classify the spherical actions of a semisimple algebraic group G on the products of the form $G/H \times G/P$, where $H \subset G$ is a symmetric subgroup, $P \subset G$ is a parabolic subgroup.

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1 Introduction

Let G be a semisimple simply connected algebraic group (defined over $\mathbb{C}$) and let $P_1, \dots, P_k \subset G$ be a list of parabolic subgroups. The product of corresponding partial flag varieties,

$$X = G/P_1 \times \cdots \times G/P_k \tag{1}$$

has a natural G variety structure via diagonal action. Determining when X is a spherical G variety carries important information for representation theory, in particular for understanding multiplicity free representations of G. We briefly summarize what is known.

For simplicity we assume that G is simple and simply connected. Let $B = UT \subset G$ be a Borel subgroup with maximal unipotent subgroup $U \subset B$ and a maximal torus $T \subset B$ that normalizes U.

Let Φ be the root system determined by (G, T) let $\Delta = \{\alpha_1, \dots, \alpha_n\} \subset \Phi$ denote its system of simple roots relative to B and let $\omega_1, \dots, \omega_n$ denote the associated fundamental dominant weights, which form a basis for the weight lattice. In particular, for each $j = 1, \dots, n$, there exists an irreducible representation of G, denoted by $V(\omega_j)$, and a line $\ell \subset V(\omega_j)$ such that the stabilizer of $\ell \in \mathbb{P}(V(\omega_j))$ is a maximal parabolic subgroup $P_j \subset G$, whence the partial flag variety G/P_j has a canonical embedding $G/P_j \hookrightarrow \mathbb{P}(V(\omega_j))$.

Let $\mathscr{C}_j \subset V(\omega_j)$ denote the cone over G/P_j and set $\mathscr{C}_{i,j} := \mathscr{C}_i \times \mathscr{C}_j$. It is shown in [12] by Littelmann that, for the diagonal action of U, invariant functions on $\mathbb{C}[\mathscr{C}_{i,j}]$ is a polynomial ring and there is an explicit list of all pairs (ω_i, ω_j) such that $\mathbb{C}[\mathscr{C}_{i,j}]^U$ is freely generated by elements of linearly independent weights. This amounts to classification of products of grassmannians of the form $G/P_i \times G/P_j$ such that the diagonal action of G is spherical.

In [13], for group $G = SL(n)$ and in [14] for $G = Sp(2n)$, a classification of parabolic subgroups $P_1, \dots, P_k$ such that the product $X = G/P_1 \times \cdots \times G/P_k$ is a spherical G variety via diagonal action is found by Magyar, Weyman, and Zelevinski. It turns out, according to [13], for X to be spherical, it has to hold true that $k \leq 3$. Among other things, in [16], the list of all parabolic subgroups $P_i \subset G$, $i = 1, 2$, where G is an arbitrary semisimple algebraic group and $G/P_1 \times G/P_2$ is G-spherical is given by Stembridge.

Let us call a partial flag variety of the form G/P, a G-flag variety. Of course, there is no particular reason for confining ourselves to the products of G-flag varieties only. Indeed, there are very important variations of this theme. Let $K \subset G$ be a symmetric subgroup of a connected simple algebraic group. (A subgroup $K \subset G$ is called symmetric if there exists an involutory automorphism $\theta : G \to G$ such that $K = \{g \in G : \ \theta(g) = g\}$.) Let B_K be a Borel subgroup of K and let $P \subset G$ be a parabolic subgroup. The natural related question is

Question: What are the conditions on (G, K, P) so that the diagonal action of G on the "$G \times K$-flag variety" $G/P \times K/B_K$ is a spherical action?

It turns out that a classification of such triplets (G, K, P) is equivalent to the classification of K-spherical G-flag varieties. A proof of the equivalence, as well as classification of these triplets is given in [8]. More recently, towards the goal of better understanding vector valued orthogonal polynomials, van Pruijssen [18] has extended the classification in [8] to the case when H is an arbitrary connected reductive subgroup. See also [15].

Motivated by these works, in this paper, we investigate a related problem. More precisely, we analyze triplets of the form (G, H, P), where G is a connected semisimple group, $H = G^\theta \subset G$ is a symmetric subgroup, and P is a parabolic subgroup of G such that the diagonal action of G on $G/H \times G/P$ is spherical. Before we state our result in its most general form we briefly set up the terminology.

Let (T, B) be a θ-split pair of maximal torus T and a Borel subgroup $B \subset G$ containing T. It is well known that there is always such a pair, [19]. Let $\Delta = \Delta(G, B, T)$ denote the subset of simple roots determined by (T, B). The automorphism θ induces an involution on the root system $\Phi = \Phi(G, T)$.

Theorem 1.1. Let (B, T) be a θ-split pair from G, $I \subset \Delta(G, B, T)$ be a subset of the simple roots, and let $P = P_I \subset G$ denote the corresponding standard parabolic subgroup. Finally, let $J = \{\alpha \in \Delta : \ \theta(\alpha) = \alpha\}$ be the set of θ-fixed simple roots. Then the product $G/P_I \times G/H$ is a spherical G-variety if and only if $G/P_I \times G/P_J$ is a spherical G-variety.

As a result of our main result we are able to give, in Section 7, a precise list of triplets (G, H, P) when G is a simple linear group. In the proof of our main result we make use of the "wonderful compactification" of the symmetric variety G/H. In fact, our general result holds in a much more general setting where $H \subset G$ is an arbitrary spherical subgroup (so that its wonderful embedding exists). The details of this progress will be presented in an upcoming article.

2 Preliminaries

2.1 G orbits in diagonal actions.

Let $K_i \subset G$, $i = 1, 2$ be two closed subgroups and set $Z_1 := G/K_1$ and $Z := G/K_2$ (there is no typo). The groups $K_1 \subset G$ act on $G \times Z$ freely as follows

$$g \cdot (g_1, x) = (g_1 g^{-1}, gx), \qquad g \in G \text{ and } (g_1, x) \in G \times Z. \tag{2}$$

We denote the quotient of the action of K_1 by $G \times^{K_1} Z$. The map

$$\varphi : G \times^{K_1} Z \to Z_1 \times Z, \qquad [(g, qK_2)] \mapsto (g^{-1}K_1, qK_2) \tag{3}$$

is a G-equivariant isomorphism. See [17, Lemma 2.3]. We have a remark in order.

Remark 2.1. 1. Z is a subvariety of $Z_1 \times Z$ via its natural embedding

$$Z \hookrightarrow \{e\} \times Z,$$

where e denotes the identity coset in $1_G K_1$ in G/K_1. Using (3), we see that $Z = \{[(1_G, yK_2)] \in G \times^{K_1} Z : \; y \in G\}$. From now on we identify Z by this copy inside $G \times^{K_1} Z$.

2. Any G-subvariety $Y \subseteq G \times^{K_1} Z$ is G-isomorphic to $G \times^{K_1} (Y \cap Z)$.

Now we assume that K_1 is a parabolic subgroup of G and let $Z' \subset Z$ be a K_1-stable closed subvariety.

Lemma 2.2. The $GZ' \subseteq Z_1 \times Z$ is a closed G-subvariety of $Z_1 \times Z$.

Proof. This is immediate from Remark 2.1 and [17, Proposition 2.7]. □

We conclude this subsection by spelling out a simple but important remark that is essentially a consequence of Remark 2.1.

Remark 2.3. Let $K \subset G$ be a subgroup and let Z be a G-variety. There is a one-to-one correspondence between G orbits in $G \times^K Z$ and K orbits in Z. In particular, if $K = B$ is a Borel subgroup of G, then it follows from (3) that $G/B \times Z$ is a spherical G-variety if and only if Z is a spherical B-variety.

2.2 Symmetric varieties.

Involutive automorphisms of algebraic groups (defined over a perfect field) are well understood; see [9] and the references therein. Since it is going to be useful in the sequel we are going to mention the crude classification, in terms of (unlabeled) Satake diagrams, of the symmetric varieties of a simple, simply connected algebraic group G that is defined over $\mathbb{C}$.

Let θ be an involutory automorphism of G. The Satake diagram of the pair (G, θ) is essentially an enrichment of the Dynkin diagram of the simple root system $\Delta = \Delta(G, B, T)$, where (T, B) is a θ-split pair. For such a choice, θ acts on the associated root system $\Phi = \Phi(G, T)$. To build the Satake diagram of (G, θ), certain nodes of the Dynkin diagram of Φ are colored according to the following rule: If $\alpha \in \Delta$ vanishes on the Lie algebra $\mathrm{Lie}(S)$ of the maximal θ-split substorus $S \subset T$, then the node corresponding to α is colored black. The simple roots that have nonzero restriction to $\mathrm{Lie}(S)$ correspond to nodes that are not colored; if two simple roots with non-colored nodes have the same restriction to $\mathrm{Lie}(S)$, then the corresponding nodes are joined by a double-pointed arrow.

3 A criterion for sphericity

3.1 Local Structure Theorem

There is a useful way, due to Brion, Luna, and Vust [4] to see if a certain variety Y is G spherical or not.

Let Y be a normal G-variety and let y be a point from a closed G-orbit in Y. Assume $G \cdot y$ is of the form $G/\tilde{Q}$ for some parabolic subgroup $\tilde{Q}$. Let Q denote the opposite parabolic to $\tilde{Q}$, let L denote the Levi subgroup $Q \cap \tilde{Q}$ and Q^u denote the unipotent radical of Q. Then [4, Theorem 1.4] says that there exists a locally closed affine subvariety $Z \subset Y$ and $y \in Z$ such that Z is stable under the action of L. Moreover, the map $Q^u \times Z \to Y$ defined by action of Q^u on Z is an open immersion. In particular, Y is spherical as a G-variety if and only if Z is spherical as an L-variety.

Remark 3.1. If Y is smooth, then Z is isomorphic, as an L-variety, to the normal space at y of the orbit $G \cdot y$ in Y.

We are going to apply this local structure theorem in our situation.

3.2 Key observation.

To state our version of Littelmann's observation in [12], we are going use (wonderful) compactifications of symmetric varieties. However, we should mention that our initial observations apply to any equivariant embedding of G/H.

Let $\overline{G/H}$ denote an equivariant embedding of the homogenous variety G/H. Our key observation is the following.

Lemma 3.2. Let K and H be two closed subgroups of G. In this case, $G/H \times G/K$ is a spherical G-variety if and only if $\overline{G/H} \times G/K$ is a spherical G-variety.

Proof. Let X^0 and X denote $G/H \times G/H$ and $\overline{G/H} \times G/K$, respectively.

($\Rightarrow$) Let $B \subset G$ be a Borel subgroup which has an open and dense orbit in X^0. Since X^0 is open and G invariant in X, the orbit of B is open and dense in X as well.

($\Leftarrow$) If X is a spherical G-variety, then any G-stable subvariety, in particular X^0, is spherical. □

The following observation will be useful in the sequel.

Lemma 3.3. Let Y_i, $i = 1, 2$ be two G-varieties and let $y_i \in Y_i$, $i = 1, 2$ be two points. Let H and K denote the isotropy subgroups of y_1 and y_2, respectively. If $G \cdot y_i$ is closed in Y_i for $i = 1, 2$, then $K \cdot y_1$ is closed in Y_1 if and only if $H \cdot y_2$ is closed in Y_2 if and only if $G \cdot (y_1, y_2)$ is closed in $Y_1 \times Y_2$.

Proof. This is the part (ii) of [1, Lemma 3.6]. □

3.3 Wonderful compactification.

Next we briefly review the theory of wonderful compactification of a symmetric space, in particular the structure of its closed G orbit.

Theorem 3.4.

1. $\overline{G/H}$ is a smooth projective G variety.
2. The boundary $\overline{G/H} - G/H$ is a union of the form $\cup_{i=1}^{l} D_i$, where D_i's are smooth divisors with normal crossings; each D_i is a closure of a single G-orbit.
3. The closure of each G-orbit in $\overline{G/H} - G/H$ is a transversal intersection of D_i's that contain the orbit.
4. The intersection of all D_i's is the unique closed G-orbit.

Proof. See [6]. □

We have a series of remarks regarding this theorem.

1. We start with describing the closed orbit in more detail. For this purpose, we start with a θ-stable maximal torus T and we assume that it contains a maximal θ-split subtorus S. (A torus S is called split with respect to θ if $\theta(s) = s^{-1}$ for all $s \in S$.)

   We denote by Φ_0 the set of roots $\alpha \in \Phi = \Phi(G, T)$ such that $\theta(\alpha) = \alpha$ and set $\Phi_1 := \Phi - \Phi_0$. There is always a Borel subgroup B, containing T, such that the corresponding set of positive roots $\Phi^+ = \Phi^+(G, B, T)$ satisfies $\theta(\Phi^+ \cap \Phi_1) \subset -\Phi^+$.

   Let Δ denote the set of simple roots corresponding to Φ^+ and set $\Delta_1 := \Phi_1 \cap \Delta$. The system of *restricted roots* is defined as follows. Let $X^*(T)$ denote the character lattice of T. $X^*(S)$ is defined similarly. There is a natural map $r : X^*(T) \to X^*(S)$ that is defined by restriction of a character on T to S. Let $\overline{\Delta}$ denote the image of Δ_1 under r. Then the root system spanned by $\overline{\Delta}$ is called the restricted root system of the symmetric variety G/H.

We are going to be more specific about the combinatorial description of the closed G orbit, so let us mention here its geometric description. It turns out that the closed orbit is isomorphic to G/P_J, where P_J is the stabilizer subgroup of the line $\ell_\lambda \subset \mathbb{P}(V_\lambda)$, where λ is a "special regular weight" for the symmetric space G/H and V_λ is the irreducible representation of G with highest weight λ. Therefore, it is uniquely determined by the Borel subgroup B and the set of simple roots α such that $\langle \alpha, \lambda \rangle = 0$.

2. The imbedding of the closed orbit, $\iota : G/P_J \to \overline{G/H}$, induces an injective homomorphism of Picard groups:
$$\iota^* : \mathrm{Pic}(\overline{G/H}) \to \mathrm{Pic}(G/P_J).$$
Without loss of generality we identify $\mathrm{Pic}(\overline{G/H})$ with a subgroup of $\mathrm{Pic}(G/P_J)$. It is not difficult to see that for any simple restricted root $\overline{\alpha} \in J$, the corresponding character $2\overline{\alpha}$ (of S and T) extends to a character of P_J whence it defines a line bundle $\mathcal{L}_{2\overline{\alpha}}$ on G/P_J. In particular, there is a one-to-one correspondence between the boundary divisors D_i's and the elements of J.

3. It follows from the previous item that each G-orbit in $\overline{G/H}$ is uniquely determined by a subset Γ of $\overline{\Delta}$. Furthermore, there is an order reversing isomorphism between the inclusion poset of G orbit closures in $\overline{G/H}$ and the inclusion poset on the subsets of $\overline{\Delta}$. (In particular, the closed G orbit corresponds to $\overline{\Delta}$.)

On a related note, for each subset $\Gamma \subset \overline{\Delta}$, there is a corresponding (standard) parabolic subgroup that we denote by $P_{\Omega(\Gamma)}$. Here, $\Omega(\Gamma)$ stands for the subset of Δ consisting of simple roots that does not restrict to an element of Γ. In other words,
$$\Omega(\Gamma) := \{\alpha \in \Delta : \ r(\alpha) \notin \Gamma\}.$$
Consequently, $\Omega(\overline{\Delta}) = \{\alpha \in \Delta : \ \overline{\alpha} \notin \overline{\Delta}\} = \{\alpha \in \Delta : \ \theta(\alpha) = \alpha\} = J$.

Let us fix a subset $\Gamma \subset \overline{\Delta}$ and denote $\Omega(\Gamma)$ simply by Ω. Let L_Ω denote the (standard) Levi subgroup of P_Ω, M denote its derived subgroup. Then θ induces an automorphism on M of order at most 2. By abusing the notation we denote this automorphism by θ as well. Then, the orbit O_Γ and its closure are of the form
$$O_\Gamma \cong G \times_{P_\Omega} M/M^\theta \ \text{ and } \ \overline{O_\Gamma} \cong G \times_{P_\Omega} \overline{M/M^\theta}.$$

4. Let $O = O_\Gamma$ be a G-orbit. Since $\overline{O}$ is the transversal intersection of boundary divisors the normal bundle to $\overline{O}$ in $\overline{G/H}$ has a precise description:
$$\mathrm{N}_{\overline{O},\overline{G/H}} := T_{\overline{O}}(\overline{G/H}) = \bigoplus_{\overline{\alpha} \in \Gamma} \mathcal{L}_{2\overline{\alpha}}|_O. \tag{4}$$

5. We describe the action of maximal torus on $\mathrm{N}_{\overline{O},\overline{G/H}}$ in the case when the orbit O contains a torus fixed point. There is a decomposition of the parabolic subgroup P_Ω in the form
$$P_\Omega = MT_\Omega \rtimes R_u(P_\Omega),$$

where $T_\Omega \subset T$ is the identity component of the intersection of the kernels of the roots $\alpha \in \Omega$, and $R_u(P_\Omega)$ is the unipotent radical of P_Ω. Then T has, up to isogeny, a corresponding decomposition

$$T = T_\Omega \cdot T'$$

for some complementary subtorus T'. Accordingly, there is a decomposition of the character lattices:

$$X^*(T) = X^*(T_\Omega) \oplus X^*(T'). \tag{5}$$

For $\overline{\alpha} \in \Gamma$, let $\alpha \in \Delta$ denote an element such that $r(\alpha) = \overline{\alpha}$. Now decompose 2α according to splitting (5):

$$2\alpha = \mu + \gamma. \tag{6}$$

In [7, Proposition 2.4], it is shown that if x is a T-fixed point contained in O and $w \in W/W_\Omega$ is an element corresponding to x, then μ is actually a character of T, not just of T_Ω, and moreover, T acts on the fiber $\mathcal{L}_{2\overline{\alpha}}(x)$ at x of the line bundle $\mathcal{L}_{2\overline{\alpha}}$ by the character $-w \cdot \mu$.

4 The slice action.

We are ready to state our version of Littelmann's observation. To this end, we assume that P_I is a standard parabolic subgroup for some subset $I \subseteq \Delta(G, B, T)$, where (T, B) is a θ-split pair.

Let $y_2 := e_{P_J}$ denote the identity coset in the closed G-orbit G/P_J of $\overline{G/H}$ and let $y_1 := e_P$ denote the identity coset in G/P_I. Let $\mathcal{O}$ denote the orbit $G\cdot(y_1, y_2) \subset \overline{G/H} \times G/P_I$. Clearly, the P_I-orbit $P_I e_{P_J}$ in G/P_J (or, P_J-orbit $P_J e_{P_I}$ in G/P_I) is closed. It follows from Lemma 3.3 that $\mathcal{O}$ is closed in $G/P_I \times \overline{G/H}$. In particular,

$$\mathcal{O} \subset G/P_I \times G/P_J \subset G/P_I \times \overline{G/H} \tag{7}$$

holds true. The isotropy group of (y_1, y_2) in G is $P_I \cap P_J$ which is equal to the standard parabolic subgroup $P_{I\cap J}$. Note also that the (standard) Levi subgroup L_J of P_J is stable under θ. The same property hold true for $P_{I\cap J}$; the standard Levi subgroup of $P_{I\cap J}$ is $L_{I\cap J}$ and it is stable under θ.

Next, we are going to investigate the action of $L_{I\cap J}$ on the normal space to $\mathcal{O} \subset G/P_I \times \overline{G/H}$ at (y_1, y_2). We start with an observation whose proof is evident.

Lemma 4.1. Let $Z \subset Y$ be two smooth subvarieties of a smooth variety X. Then the normal bundle of Z in X is $T_Z(Y) \oplus T_Y(X)$.

It follows from (7) and Lemma 4.1 that the normal space of $\mathcal{O} \subset G/P_I \times \overline{G/H}$ at (y_1, y_2) is given by

$$T_{\mathcal{O}}(G/P_I \times G/P_J)|_{(y_1,y_2)} \ \oplus \ T_{G/P_I\times G/P_J}(G/P_I \times \overline{G/H})|_{(y_1,y_2)}. \tag{8}$$

Remark 4.2. 1. The decomposition in (8) is equivariant with respect to the action of $L_{I\cap J}$. Therefore, $G/P_I \times \overline{G/H}$ is a spherical G-variety if and only if the action of $L_{I\cap J}$ on both of the spaces $T_{\mathcal{O}}(G/P_I \times G/P_J)|_{(y_1,y_2)}$ and $T_{G/P_I\times G/P_J}(G/P_I \times \overline{G/H})|_{(y_1,y_2)}$ are spherical. The sphericity of the first action amounts to the sphericity of the G-variety $G/P_I \times G/P_J$.

2. The second summand in (8) is already computed in the previous section:

$$\begin{aligned} T_{G/P_I\times G/P_J}(G/P_I \times \overline{G/H})|_{(y_1,y_2)} &= T_{G/P_J}(\overline{G/H})|_{y_2} \\ &\cong \bigoplus_{\alpha\in\overline{\Delta}} \mathcal{L}_{2\overline{\alpha}}|_{G/P_J}(y_2). \end{aligned} \tag{9}$$

5 Multiplicity free representations

Recall that a representation V of G is called multiplicity free if the induced action of G on the polynomial functions $k[V]$ decomposes into a direct sum of irreducible G-modules and no irreducible representation appears more than once in this decomposition. There are various other characterizations of this property:

- V is a spherical G-variety;
- The subalgebra of B-invariant rational functions on V is one dimensional.

The classification of such representations has a long history. When the action of G on V is irreducible, the initial classification is given by Kac [10]. In [3], Brion extended the classification of Kac to all multiplicity free representations of simple groups. Finally, Leahy [11] and Benson-Ratcliff [2] independently classified all multiplicity free representation of connected reductive groups.

Remark 5.1. The results of Leahy [11] and Benson-Ratcliff [2] are stated with the assumption that the underlying field of definitions is $\mathbb{C}$, however, by the work of [5], we know that they are valid over arbitrary algebraically closed field of prime characteristic.

6 First approximation and the main result

We go back to our notation from Section 4. Let (Φ, Δ) denote the pair of root system and its basis associated with a split pair (T, B) in G and let $\Phi_0 = \{\alpha \in \Phi :\ \theta(\alpha) = \alpha\}$. We assume that the system of positive roots Φ^+ has the following property:

$$\Phi^+ - \Phi_0 = \{\alpha \in \Phi :\ \theta(\alpha) \in -\Phi^+\}.$$

As before, we set

$$J := \{\alpha \in \Delta :\ \overline{\alpha} \notin \overline{\Delta}\} = \{\alpha \in \Delta :\ \theta(\alpha) = \alpha\}.$$

Then G/P_J is the closed orbit in the wonderful compactification $\overline{G/H}$.

Proposition 6.1. The product $G/P_I \times G/H$ is a spherical G-variety if and only if

1. the G-variety $G/P_I \times G/P_J$ is spherical, and
2. the linear action of the standard Levi subgroup $L_{I\cap J}$ on

$$\mathrm{N} = \bigoplus_{\alpha\in\overline{\Delta}} \mathcal{L}_{2\overline{\alpha}}|_{G/P_J}(e_{P_J}) \tag{10}$$

is multiplicity free, hence it appears in [2, Section 6.5].

Here $\mathcal{L}_{2\overline{\alpha}}$ is the line bundle on G/P_J that is determined by the simple root $2\overline{\alpha}$, and $\mathcal{L}_{2\overline{\alpha}}(e_{P_J})$ denotes its fiber at the identity coset.

Proof. The proof follows from Remark 4.2 and the above considerations. □

Remark 6.2. There is an obvious obstruction for sphericity of the action of $L_{I\cap J}$ on (10), that is

$$\dim L_{I\cap J} \geq |\overline{\Delta}|. \tag{11}$$

Proposition 6.1 tells us that there are two steps towards a classification of G-spherical products of the form $G/P_I \times G/H$. In the first step one needs to know $G/P_I \times G/P_J$ is spherical with respect to diagonal action of G and in the second step one needs to know the multiplicity-free actions of $L_{I\cap J}$ on N, the linear space in (10). The first step is easily determined by calculating the set of restricted roots, for example by looking at the Satake diagram of (G, H), and by the list of Stembridge that is provided in the appendix to our paper. Our goal in the rest of this section is to show that the second condition is vacuous.

We recall the action of T on $\mathcal{L}_{2\overline{\alpha}}|_{G/P_J}(e_{P_J})$. We have the Levi decomposition $P_J = MT_J \rtimes R_u(P_J)$, where $T_J \subset T$ is the identity component of the intersection of the kernels of the roots $\alpha \in J$, and $R_u(P_J)$ is the unipotent radical of P_J. Then T has, up to isogeny, a corresponding decomposition $T = T_J \cdot T'$ for some complementary subtorus T'. Accordingly, there is a decomposition of the character lattices: $X^*(T) = X^*(T_J) \oplus X^*(T')$. For $\overline{\alpha} \in \overline{\Delta}$, let $\alpha \in \Delta$ denote an element such that $r(\alpha) = \overline{\alpha}$. Then $2\alpha = \mu + \gamma$ for some $\mu \in X^*(T_J)$ and $\gamma \in X^*(T')$ such that μ extends to a character of T. Moreover, T acts on the fiber $\mathcal{L}_{2\overline{\alpha}}(x)$ at x by the character $-\mu$. Note that μ depends on $\overline{\alpha}$, not on α, therefore, each such character of $X^*(T_J)$ appears at most once in N. In other words, the action of $L_{I\cap J}$ is multiplicity free and the sphericity is guaranteed by the inequality (11). Therefore, we have our main general result as follows.

Theorem 6.3. Let (T, B) be a θ-split pair in a semisimple algebraic group G, let $I \subset \Delta$ be a subset of the simple roots determined by (T, B), and let $P = P_I \subset G$ denote the corresponding standard parabolic subgroup. Finally, let $J = \{\alpha \in \Delta : \theta(\alpha) = \alpha\}$ be the set of θ-fixed simple roots. Then the product $G/P_I \times G/H$ is a spherical G-variety if and only if $G/P_I \times G/P_J$ is a spherical G-variety.

7 Precision

For the rest of this section, in addition to our assumptions in Section 6 we assume further that G is a simple algebraic group. Our justification for this assumption is that if G is simply connected, connected, and semi-simple, then any symmetric pair (G, H) is a direct product of symmetric pairs (G', H') with G' simple, or G is a doubled group $G = G' \times G'$ and H is the diagonal copy of G' in G. It follows that the product space $G/P_I \times G/H$ splits accordingly.

In this section we are going to compute the pairs (I, J), where $G/P_I \times G/P_J$ is a spherical G variety and G/P_J is the closed orbit in $\overline{G/H}$. We do this analysis, case-by-case, for each symmetric variety G/H.

7.1 $(G, H) = (G \times G, diag(G))$.

Let (T, B) be any pair of maximal torus and a Borel subgroup from G. Let Δ denote the corresponding set of simple roots. For simplicity we denote $G \times G$ by G_1. The canonical involution on G_1 is given by $\theta(g, h) = (h, g)$, $g, h \in G$. Then $S := \{(t, t^{-1} : t \in T\} \subset G_1$ is a maximal θ-split torus and $T_1 := T \times T \subset G_1$ is a maximal torus containing S. The root system of (G_1, T_1) is the disjoint union of two copies of $\Phi = \Phi(G, T)$ and $\Delta \sqcup \Delta$ forms a set of simple roots for $\Phi(G_1, T_1)$. The induced involution on $\Phi(G_1, T_1)$ sends a root from the first copy of Φ to the same element in the other copy. Thus, there is no θ-fixed root and the closed G_1 orbit is isomorphic to $G/B \times G/B$.

Parabolic subgroups of G_1 are of the form $P \times Q$, where P and Q are parabolic subgroups of G. Now, fix a standard parabolic subgroup $P_1 = P \times Q$. Note that $G_1/P_1 \cong G/P \times G/Q$. By Theorem 6.3, we have that $G_1/G \times G_1/P_1$ is a spherical G_1-variety if and only if

$$(G/B \times G/B) \times (G/P \times G/Q) \cong (G/B \times G/P) \times (G/B \times G/Q)$$

is G_1-spherical. Since we are using diagonal action of $G_1 = G \times G$, this is equivalent to both of $G/B \times G/P$ and $G/B \times G/Q$ being G-spherical. But since $B \subset P \cap Q$, neither G/Q nor G/P is B-spherical, therefore, by Remark 2.3 we see that $G_1/G \times G_1/P_1$ is never a spherical G_1-variety.

7.2 $(G, H) = (SL(n), Spin(n))$.

The maximal torus of diagonal matrices and the Borel subgroup of upper triangular matrices in $G = SL(n)$ form a split pair for G/H. The Satake diagram of this pair is depicted in Figure 1. (For the remaining cases, instead of explicitly stating the θ-split pair (T, B) we are going to draw the Satake diagram only.)

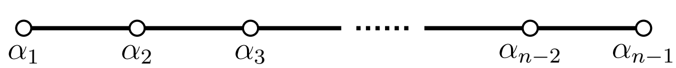

Figure 1: The Satake diagram of $(SL(n), Spin(n))$.

It is clearly the case that $J = \emptyset$, whence the closed orbit of $\overline{G/H}$ is G/B. It follows from Remark 2.3 that for $G/H \times G/P_I$ is a non-spherical G-variety.

7.3 $(G, H) = (SL(2n), Sp(2n))$.

The Satake diagram of this pair is depicted in Figure 2.

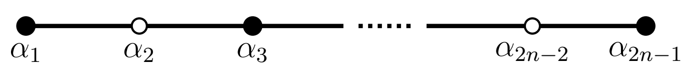

Figure 2: The Satake diagram of $(SL(2n), Sp(2n))$.

Thus, $J = \{\alpha_1, \alpha_3, \dots, \alpha_{2n-1}\}$. In this case, it follows from items 1(a)–1(d) of the appendix that the possibilities are

1. $I^c = \emptyset$, $\{1\}$, $\{2n-1\}$;
2. $n = 2$, $I^c = \{2\}$ and $J^c = \{1, 3\}$;

7.4 $(G, H) = (SL(p+q), S(GL(p) \times GL(q)))$, where $p < q$.

Let n denote the sum $p + q$. The Satake diagram of this pair is depicted in Figure 3.

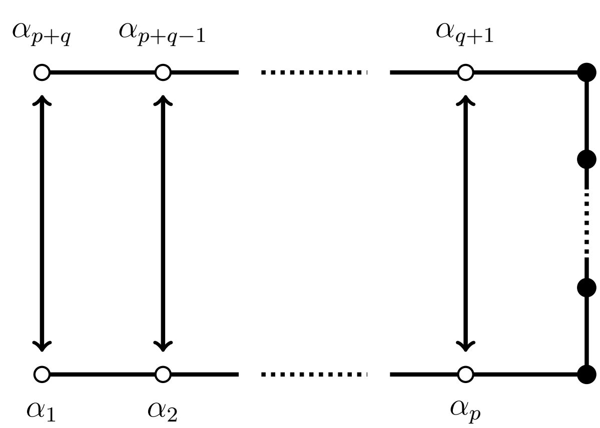

Figure 3: The Satake diagram of $(SL(p+q), S(GL(p) \times GL(q)))$.

Then $J = \{\alpha_{p+1}, \alpha_{p+2}, \dots, \alpha_q\}$. It follows from 1(a)–1(d) of Appendix that

1. $I^c = \emptyset$, $\{1\}$, $\{n\}$;
2. $p = 1$, $q = n-1$; $I^c = \{2\}$ or $\{n-1\}$ and $J^c = \{1, n\}$.

7.5 $(G, H) = (Sp(2n), GL(n))$.

In this case, the Satake diagram is as in Figure 4.

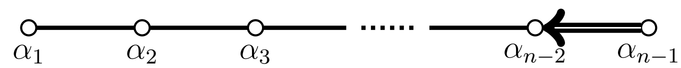

Figure 4: The Satake diagram of $(Sp(2n), GL(n))$.

Then $J = \emptyset$, hence $G/P_I \times G/H$ is a non-spherical G-variety.

7.6 $(G, H) = (Spin(2n), GL(n))$, n even.

The Satake diagram of this pair is depicted in Figure 5.

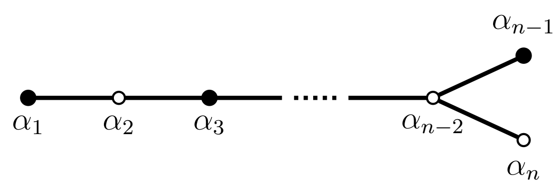

Figure 5: The Satake diagram of $(Spin(2n), GL(n))$ for n even.

Then $J = \{\alpha_1, \alpha_3, \ldots, \alpha_{n-3}, \alpha_{n-1}\}$. By looking at the items 4(a)–(c) of Appendix, we see that the only possibility is when $n = 4$; $I^c = \{3\}$ and $J^c = \{2, 4\}$.

7.7 $(G, H) = (Spin(2n), GL(n))$, n odd.

The Satake diagram of this pair is depicted in Figure 6.

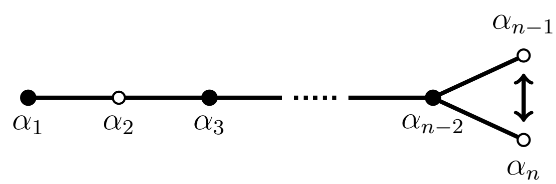

Figure 6: The Satake diagram of $(Spin(2n), GL(n))$ for n odd.

Then $J = \{\alpha_1, \alpha_3, \ldots, \alpha_{n-2}\}$. By looking at the items 4(a)–(c) of Appendix, we see that the only possibility is when $I^c = \{n\}$.

7.8 $(G, H) = (Spin(p+q), S(O(p) \times O(q)))$ with $0 < p < q$ and $p+q$ odd.

Let $n = p + q$ and suppose $n = 2l + 1$ for some positive integer l. In this case, the Satake diagram of this pair is depicted in Figure 7.

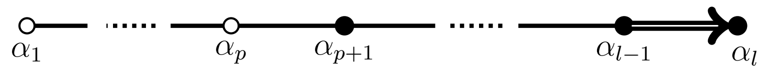

Figure 7: The Satake diagram of $(Spin(p+q), S(O(p) \times O(q)))$ with $0 < p < q$ and $p+q$ odd.

Then $J = \emptyset$, hence $G/P_I \times G/H$ is a non-spherical G-variety.

7.9 $(G, H) = (Spin(p+q), S(O(p) \times O(q)))$ with $0 < p < q$ and $p+q$ even.

Let n denote the even number $p + q = 2l$ (for some positive integer l). If $p = l - 1$, then the corresponding Satake diagram is as depicted in Figure 8.

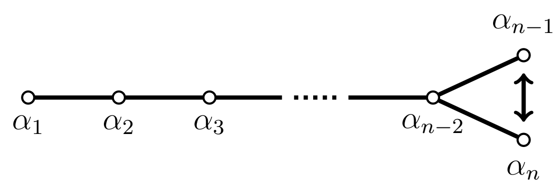

Figure 8: $0 < p < q$, $2l = p + q$, and $p = l - 1$.

Then $J = \emptyset$, hence $G/P_I \times G/H$ is a non-spherical G-variety. If $p < l - 1$, then the corresponding Satake diagram is depicted in Figure 9.

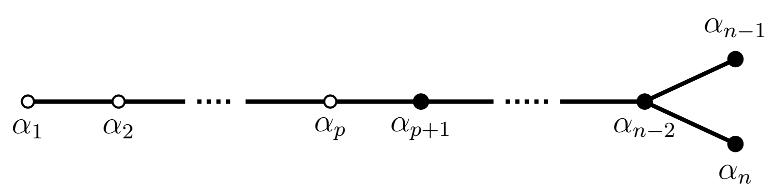

Figure 9: $0 < p < q$, $2l = p + q$, and $p < l - 1$.

Then $J = \{\alpha_{p+1}, \dots, \alpha_n\}$. It follows from items 4(a)–(c) of Appendix that the possibilities for I and J are

1. $I^c = \{1\}$ and $J^c = \{1\}$;
2. $I^c = \{n\}$, or $I^c = \{n - 1\}$ and $J^c \subseteq \{1, 2\}$.

7.10 $(G, H) = (Sp(2n), Sp(2p)\times Sp(2q))$ with $1 \leq p < q \leq n-1$, $p+q = n$.

If $2p < n$, then the Satake diagram of this pair is depicted in Figure 10.

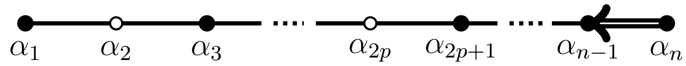

Figure 10: The Satake diagram of $(Sp(2n), Sp(2p) \times Sp(2q))$ with $2p < n$.

Then $J = \{\alpha_1, \alpha_3, \alpha_{2p-1}, \alpha_{2p+1}, \alpha_{2p+2}, \dots, \alpha_n\}$. It follows from item 3 of Appendix that we must have

$$p = 1;\ I^c = \{1\}, \quad \text{and } J^c = \{2\}.$$

If $2p = n$, then the Satake diagram of this pair is depicted in Figure 11.

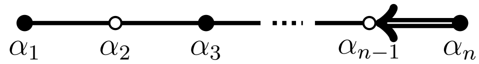

Figure 11: The Satake diagram of $(Sp(2n), Sp(2p) \times Sp(2q))$ with $2p = n$.

Then $J = \{\alpha_1, \alpha_3, \dots, \alpha_n\}$ hence the cardinality of its complement is at least 2 since $n \geq 3$. Therefore, it follows from item 3 of Appendix that there is no I and J such that $G/H \times G/P_I$ is G-spherical.

7.11 $(G, H) = (E_6, SL(6) \times SL(2))$.

The Satake diagram of $(E_6, Spin(5)\times\mathbb{C}^*))$. is as in Figure 12. Then $J = \emptyset$, hence $G/P_I\times G/H$

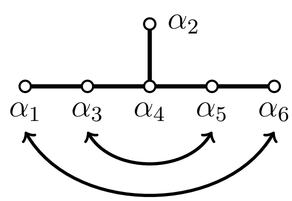

Figure 12: Satake diagram of $(E_6, SL(6) \times SL(2))$.

is a non-spherical G-variety.

7.12 $(G, H) = (E_6, Spin(5) \times \mathbb{C}^*)$.

The Satake diagram of $(E_6, Spin(5) \times \mathbb{C}^*))$. is as in Figure 13.

Then $J = \{\alpha_3, \alpha_4, \alpha_5\}$ and it follows from 5th item of the appendix that there is no spherical product of the form $G/H \times G/P_I$.

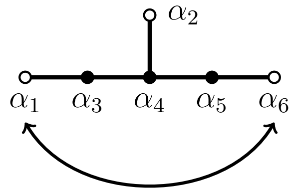

Figure 13: Satake diagram of $(E_6, Spin(5) \times \mathbb{C}^*)$.

7.13 $(G, H) = (E_6, Sp(4))$.

The Satake diagram of $(E_6, Sp(4))$. is as in Figure 14. Then $J = \emptyset$, hence $G/P_I \times G/H$ is

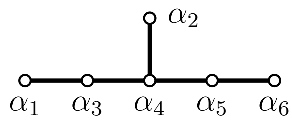

Figure 14: Satake diagram of $(E_6, Sp(4))$.

non-spherical G-variety.

7.14 $(G, H) = (E_6, F_4)$.

The Satake diagram of (E_6, F_4). is as in Figure 15. Then $J = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ and it follows

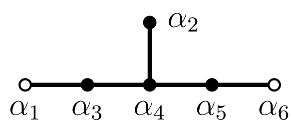

Figure 15: Satake diagram of (E_6, F_4).

from 5th item of the appendix that the possibilities are

$$I^c = \{1\}, \text{ or } \{6\} \text{ and } J^c = \{1, 6\}.$$

7.15 $(G, H) = (E_7, SL(7))$.

The Satake diagram of $(E_7, SL(7))$ is as in Figure 16. Then $J = \emptyset$, hence $G/P_I \times G/H$ is non-spherical G-variety.

7.16 $(G, H) = (E_7, Spin(6) \times SL(2))$.

The Satake diagram of $(E_7, Spin(6) \times SL(2))$ is as in Figure 17. Then $J = \{\alpha_1, \alpha_3, \alpha_4, \alpha_6\}$. It follows from item 6 of Appendix that there is no spherical product of the form $G/H \times G/P_I$.

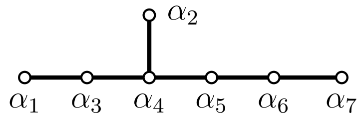

Figure 16: Satake diagram of $(E_7, SL(7))$.

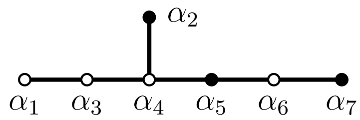

Figure 17: Satake diagram of $(E_7, Spin(6) \times SL(2))$.

7.17 $(G, H) = (E_7, E_6 \times \mathbb{C}^*)$.

The Satake diagram of $(E_7, E_6 \times \mathbb{C}^*)$ is as in Figure 18. Then $J = \{\alpha_1, \alpha_6, \alpha_7\}$. It follows

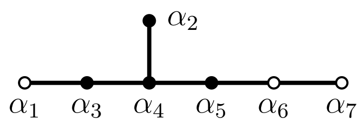

Figure 18: Satake diagram of $(E_7, E_6 \times \mathbb{C}^*)$.

from item 6 of Appendix that there is no spherical product of the form $G/H \times G/P_I$.

8 Appendix: The list of spherical double cones.

Let G be a simple linear algebraic group. We label the nodes of Dynkin diagram of G as in Table 19. This labeling is opposite to the one that is used by Stembridge [16] in the cases of type B,C, and D.

Below is the list, due to Stembridge [16], of products $G/P_I \times G/P_J$ which are G-spherical with respect to the diagonal action of G. For each item, we use the indices of simple roots on the corresponding Dynkin diagram as depicted in Figure 19.

1. $SL(n+1)$:
   (a) $I^c = \emptyset$, $\{1\}$, $\{n\}$;
   (b) $I^c = \{2\}$ or $\{n-1\}$ and $|J^c| = 2$;
   (c) $|I^c| = |J^c| = 1$;
   (d) $|I^c| = 1$ and $J^c = \{1, j\}, \{j, j+1\}$, or $\{j, n\}$ with $1 < j < n$.
2. $Spin(2n+1)$: Assuming I and J are proper, we have

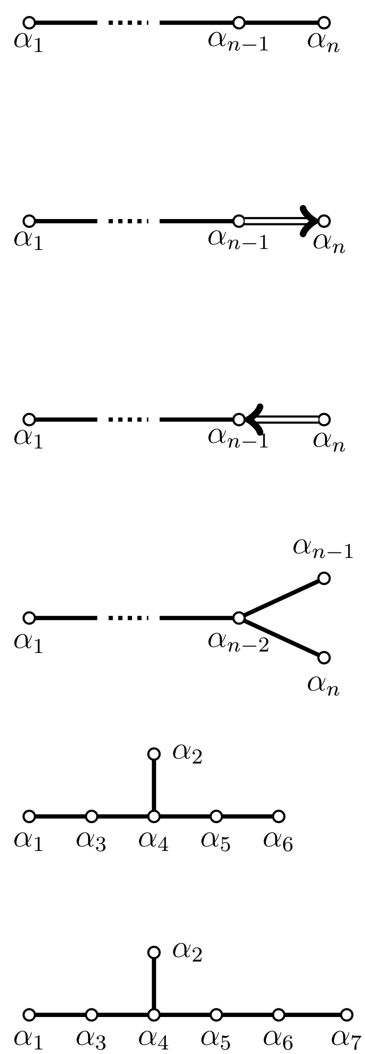

Figure 19: A labeling of the relevant Dynkin diagrams

(a) $I^c = \{1\}$ and $|J^c| = 1$;

(b) $I^c = J^c = \{n-1\}$.

3. $Sp(2n)$: Assuming I and J are proper, we have

   (a) $I^c = \{1\}$ and $|J^c| = 1$;

   (b) $I^c = J^c = \{n\}$.

4. $Spin(2n)$: Assuming that $n \geq 4$, I and J are proper, we have

   (a) $I^c = \{1\}$ and $J^c = \{i\}, \{i, n\}$, or $\{i, n-1\}$ $(1 \leq i \leq n)$;

   (b) $I^c = \{n\}$, or $I^c = \{n-1\}$ and $J^c \subsetneq \{n, n-1, 1\}$, $J^c \subseteq \{1, 2\}$, or $J^c = \{3\}$;

   (c) ($n = 4$ only) $I^c = \{4\}$, $J^c = \{2, 3\}$ or $I^c = \{3\}$ and $J^c = \{2, 4\}$.

5. E_6:

   (a) $I^c = \{1\}$, or $\{6\}$ and $J^c = \{i\}$ $(i \neq 4)$ or $J^c = \{1, 6\}$.

6. E_7:

   (a) $I^c = \{7\}$, or $\{6\}$ and $J^c = \{1\}, \{2\}$, or $\{7\}$.

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