The rook monoid is lexicographically shellable

Mahir Bilen Can

October 5, 2013

Abstract

The rook monoid R_n is the finite monoid consisting of $n \times n$, 0/1 matrices with at most one 1 in each row and column. It is known that R_n parametrizes $B \times B$ orbits on $n \times n$ matrices, where B is the Borel subgroup of invertible upper triangular matrices. The Bruhat-Chevalley-Renner ordering on R_n is the partial order arising from the inclusion order on these Borel orbit closures. In this manuscript, generalizing the well-known EL-shellability of the symmetric group of permutations, we show that R_n is an EL-shellable poset.

1 Introduction

For a positive integer n, let $[n]$ denote the finite set $\{1, \dots, n\}$. Let P be a finite graded poset of rank n with minimum and maximum elements denoted by $\hat{0}$ and $\hat{1}$, respectively, and let $C(P)$ denote the set of pairs $(x, y) \in P \times P$ such that y covers x. Recall that P is called *lexicographically shellable*, if there exists a map $f : C(P) \to [n]$ such that

1. in every interval $[x, y] \subseteq P$ there exists a unique unrefinable chain $\mathfrak{c} : \; x = x_0 < x_1 < \cdots < x_{k+1} = y$ such that $f(x_i, x_{i+1}) \leq f(x_{i+1}, x_{i+2})$ for all $i = 0, \dots, k-1$.
2. The sequence $f(\mathfrak{c}) := (f(x, x_1), \dots, f(x_k, y))$ of the (unique) chain $\mathfrak{c}$ of (1) is lexicographically first among all sequences of the form $f(\mathfrak{d})$, where $\mathfrak{d}$ is an unrefinable chain in $[x, y]$.

Introduced by Björner in [2], the above notion of lexicographic shellability has important topological consequences. For example, it is known that if the poset is lexicographically shellable, then $\Delta(P)$, the abstract simplicial complex of P is a shellable simplicial complex . For more on lexicographic shellability and its generalizations see the excellent survey by M. Wachs on poset topology, [19].

In this paper we are concerned with the question of shellability for R_n, the *rook monoid* of 0/1 matrices of size n with at most one 1 in each row and each column. One can view elements of R_n as non-attacking rook placements on an $n \times n$ chess board, hence the nomenclature follows. The partial ordering on R_n that we are interested in comes from the topology of the "matrix Schubert varieties," which we briefly describe now.

We fix an algebraically closed field K, and let $G = GL_n$ denote the general linear group over K. Denote by $B \subset G$ the subgroup of invertible upper triangular matrices. Let M_n be the set of all $n \times n$ matrices over K. We denote by M_n the monoid of $n \times n$ matrices. Consider the following action of $B \times B$ on M_n:

$$(x, y) \cdot g = xgy^{-1},\ g \in G,\ x, y \in B \tag{1.1}$$

For $g \in G$, the Zariski closure $\overline{B \times B \cdot g} = \overline{BgB} \subseteq M_n$ is called a *matrix Schubert variety*, and it is well known that they are parametrized by the symmetric group of permutations, denoted by S_n. Furthermore, G has the "Bruhat-Chevalley decomposition" into these orbits

$$G = \bigsqcup_{w \in S_n} BwB. \tag{1.2}$$

The action (1.1) on G extends to an action on M_n. In [13], Renner shows that the orbits of the extended action are parametrized by the rook monoid R_n, and furthermore, the analogue of (1.2) holds:

$$M_n = \bigsqcup_{r \in R_n} BrB.$$

The *Bruhat-Chevalley-Renner ordering* on R_n is defined by

$$r \leq t \iff BrB \subseteq \overline{BtB}.$$

Our main result is that the rook monoid with respect to Bruhat-Chevalley-Renner ordering is a lexicographically shellable poset. Consequently, we show that, for any interval I in R_n, the simplicial complex $\Delta(I)$ has the homotopy type of a wedge of spheres or balls.

The proof of our main result relies on an extension of EL-labeling on S_n, which is found by P. Edelman in [7]. We should mention that the collection of rook matrices of rank k ($0 \leq k \leq n$) form an interval in R_n. Of course, if $k = n$, then we recover S_n in R_n. Therefore, our result is indeed a generalization of Edelman's.

Acknowledgements. We thank Professor Lex Renner for helpful discussions on linear algebraic monoids. We are grateful to the anonymous referees and to Professor Michelle Wachs for their constructive comments and suggestions which greatly improved the quality of our paper.

2 Background

The monoid of $n \times n$ matrices is an important member of the family of varieties called algebraic monoids. To place our work appropriately in this general setting and to help the reader unfamiliar with the theory of algebraic monoids let us briefly recall the definitions and relevant combinatorial results without giving much detail. For more, see one of the following excellent sources: [14], [11], or [15].

2.1 Reductive monoids

Let G be a reductive group. Fix a maximal torus T, and a Borel subgroup B such that $T \subset B \subset G$. The *Weyl group* W associated with (G, T) is defined to be the quotient group $W = N_G(T)/T$, where $N_G(T)$ is the normalizer of T in G. In the case of $G = GL_n$ the Weyl group is isomorphic to the symmetric group S_n. The Bruhat-Chevalley order on the Weyl group W is defined by $w \leq v \iff BwB \subseteq \overline{BvB}$. It is shown by different authors that the Bruhat-Chevalley orders are lexicographically shellable (see [10], [3], and [7]).

The generalization of the Bruhat-Chevalley ordering in the realm of algebraic monoids is due to Renner, [13]. An algebraic monoid is an algebraic variety M together with an associative binary operation $m : M \times M \to M$ which is a morphism of varieties.

An interesting class of algebraic monoids is built from representations as follows. Let $\rho : G_0 \to GL(V)$ be a rational representation of a semisimple algebraic group G_0. By abuse of notation, let K^* denote the scalar matrices in the (affine) space $End(V)$ of linear transformations on V. Then, the Zariski closure $M = \overline{K^* \cdot \rho(G_0)}$ in $End(V)$ is a *reductive monoid.*

Let G be the (reductive) group of invertible elements of a reductive monoid M, and let $T \subset B \subset G$ be a maximal torus and a Borel subgroup. It is shown in [13] that reductive monoids have decompositions into double cosets of B

$$M = \bigsqcup_{r \in R} B\dot{r}B, \ \dot{r} \in \overline{N_G(T)}/T,$$

indexed by a finite monoid R, now called the *Renner monoid* of M. Here $\overline{N_G(T)}$ is the Zariski closure in M of the normalizer in G of T. The Bruhat-Chevalley-Renner ordering on R is defined as before. In the special case of the defining representation $\rho : G_0 \to GL(K^n)$ of $G_0 = SL_n$, the Renner monoid R is isomorphic to the rook monoid R_n. The Weyl group W of (G,T) forms the group of invertible elements in the Renner monoid R, and the Bruhat-Chevalley ordering on W extends to the Bruhat-Chevalley-Renner ordering on R.

There is a cross section lattice $\Lambda \subset R$ of idempotents, parametrizing the $G \times G-$orbits in M

$$M = \bigsqcup_{e \in \Lambda} GeG.$$

Furthermore,

$$R = \bigsqcup_{e \in \Lambda} WeW.$$

Let $e \in \Lambda$. In [12], Putcha shows that the subposets $WeW \subseteq R$ of $W \times W$-orbits in R are lexicographically shellable posets. It is also known that the cross section lattice $\Lambda \subseteq R$ is an (upper) semimodular lattice, hence shellable. However, showing that a Renner monoid is shellable seems to be a difficult problem.

2.2 Lexicographic shellability

We assume that P is a *graded poset* of rank n with a maximum and a minimum element, denoted by $\hat{1}$ and $\hat{0}$ respectively. Thus, all maximal chains of P have equal length n. Denote by $C(P)$ the set of covering relations

$$C(P) = \{(x,y) \in P \times P : \; y \text{ covers } x\}.$$

An *edge-labeling* on P is a map of the form $f = f_{P,\Gamma} : C(P) \to \Gamma$ for some total order Γ. The *Jordan-Hölder sequence* (with respect to f) of a maximal chain $\mathfrak{c} : x_0 < x_1 < \cdots < x_{n-1} < x_n$ of P is the n-tuple

$$f(\mathfrak{c}) = (f(x_0,x_1), f(x_1,x_2), \ldots, f(x_{n-1},x_n)) \in \Gamma^n.$$

Fix an edge labeling f, and a maximal chain $\mathfrak{c} : \; x_0 < x_1 < \cdots < x_n$. We call both of the maximal chain $\mathfrak{c}$ and its image $f(\mathfrak{c})$ *increasing*, if

$$f(x_0,x_1) \le f(x_1,x_2) \le \cdots \le f(x_{n-1},x_n)$$

holds in Γ.

Let k be a positive integer. We consider the lexicographic (total) ordering on the $k-$fold cartesian product $\Gamma^k = \Gamma \times \cdots \times \Gamma$. An edge labeling $f : C(P) \to \Gamma$ is called an *EL-labeling*, if

1. in every interval $[x, y] \subseteq P$ of rank $k > 0$ there exists a unique maximal chain $\mathfrak{c}$ such that $f(\mathfrak{c}) \in \Gamma^k$ is increasing,

2. the Jordan-Hölder sequence $f(\mathfrak{c}) \in \Gamma^k$ of the unique chain $\mathfrak{c}$ from (1) is the smallest among the Jordan-Hölder sequences of maximal chains $x = x_0 < x_1 < \cdots < x_k = y$.

A poset P is called *EL-shellable*, if it has an EL-labeling.

Remark 2.1. There are various lexicographic shellability conditions in the literature and the EL-shellability defined here is among the stronger ones. A deep relationship between EL-shellability of a Coxeter group W and the Kazhdan-Lusztig theory of the Hecke algebra associated with W is found by Dyer in [6].

2.3 The symmetric group

S_n is the set of all permutations of $[n]$. Let us represent the elements of S_n in one line notation $w = (w_1, \dots, w_n) \in S_n$ so that $w(i) = w_i$. It is well known that S_n is a graded poset with respect to Bruhat-Chevalley ordering. Let B denote, as before, the group of invertible upper triangular matrices in GL_n. Grading on S_n is given by the length function

$$\ell(w) = \dim(BwB) - \dim(B) = inv(w), \tag{2.2}$$

where

$$inv(w) = |\{(i,j) :\ 1 \leq i < j \leq n,\ w_i > w_j\}|. \tag{2.3}$$

Note that $\dim B = \binom{n+1}{2}$.

The Bruhat-Chevalley ordering on S_n is the partial order generated by the transitive closures of the following covering relations. The permutation $x = (a_1, \dots, a_n)$ is covered by the permutation $y = (b_1, \dots, b_n)$, if $\ell(y) = \ell(x) + 1$ and

1. $a_k = b_k$ for $k \in \{1, \dots, \widehat{i}, \dots, \widehat{j}, \dots, n\}$ (hat means omit those numbers),

2. $a_i = b_j$, $a_j = b_i$, and $a_i < a_j$.

An EL-labeling for S_n is constructed by Edelman [7] as follows. Let $\Gamma = [n] \times [n]$ denote the poset of pairs, ordered lexicographically: $(i, j) \leq (r, s)$ if $i < r$, or $i = r$ and $j < s$. For permutations $x, y \in S_n$ given as above, define $f(x, y) = (a_i, a_j)$. In Figure 2.1 we depict the corresponding EL-labeling for S_3.

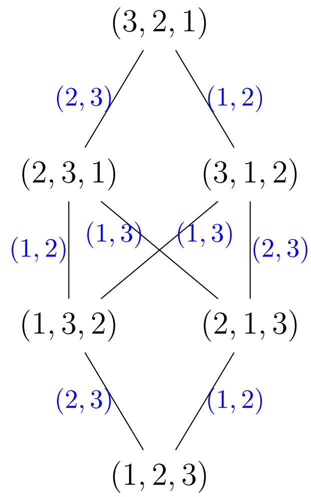

Figure 2.1: EL-labeling of S_3

Theorem 2.4. *([7]) The symmetric group S_n with Bruhat-Chevalley ordering is lexicographically shellable.*

2.4 The rook monoid

Recall from [13] that the rank function on R_n is given by

$$\ell(x) = \dim(BxB),\ x \in R_n.$$

There is a combinatorial formula for $\ell(x)$, $x \in R_n$ similar to (2.2). To explain we represent elements of R_n by n-tuples, just as we did for S_n in the previous subsection. Let $x = (x_{ij}) \in R_n$ and define the sequence $(a_1, \dots, a_n)$ by

$$a_j = \begin{cases} 0 & \text{if the } j\text{-th column consists of zeros,} \\ i & \text{if } x_{ij} = 1. \end{cases} \tag{2.5}$$

By abuse of notation, we denote both the matrix and the sequence $(a_1, \dots, a_n)$ by x. For example, the associated sequence of the partial permutation matrix

$$x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

is $x = (3, 0, 4, 0)$.

Let $x = (a_1, \dots, a_n) \in R_n$. A pair (i, j) of indices $1 \le i < j \le n$ is called a *coinversion pair* for x, if $0 < a_i < a_j$. We denote the number of conversion pairs of x by $coinv(x)$.

Example 2.6. Let $x = (4, 0, 2, 3)$. Then, the only coinversion pair for x is $(3, 4)$. Therefore, $coinv(x) = 1$.

In [4], it is shown that the dimension $\ell(x) = \dim(BxB)$ of an orbit BxB, $x \in R_n$ is given by

$$\ell(x) = (\sum_{i=1}^{n} a_i^*) - coinv(x), \text{ where } a_i^* = \begin{cases} a_i + n - i, & \text{if } a_i \neq 0 \\ 0, & \text{if } a_i = 0 \end{cases} \tag{2.7}$$

We reformulate (2.7) as follows:

Proposition 2.8. *Let $x = (a_1, \dots, a_n) \in R_n$. Then*

$$\ell(x) = \sum a_i + inv(x),$$

where $inv(x) = |\{(i, j) : \ 1 \le i < j \le n, \ a_i > a_j\}|$.

Proof. Since $coinv(x) = \sum_{i=1}^{n} c_i$, where c_i is the number of $j \in [n]$ such that $i < j$ and $0 < a_i < a_j$, we can rewrite (2.7) as follows:

$$\begin{aligned} \ell(x) &= \sum_{i=1}^{n} (a_i^* - c_i) \\ &= \sum_{i=1}^{n} (a_i + d_i^*) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} d_i^*, \end{aligned}$$

where

$$d_i^* = \begin{cases} n - i - c_i, & \text{if } a_i \neq 0, \\ 0, & \text{if } a_i = 0. \end{cases}$$

Observe now that d_i^* is the equal to the number of $j > i$ such that $a_i > a_j \ge 0$. □

The following consequence of Proposition 2.8 agrees with equation (2.2).

Corollary 2.9. *Let $w = (a_1, \dots, a_n) \in S_n$ be a permutation. Then $\ell(w) = \binom{n+1}{2} + inv(w)$.*

First concrete description of the Bruhat-Chevalley-Renner ordering on R_n is given in [9]:

Theorem 2.10. *Let $x = (a_1, \ldots, a_n)$, $y = (b_1, \ldots, b_n) \in R_n$. The Bruhat-Chevalley order on R_n is the smallest partial order on R_n generated by declaring $x \leq y$ if either*

1. *there exists an $1 \leq i \leq n$ such that $b_i > a_i$ and $b_j = a_j$ for all $j \neq i$, or*
2. *there exist $1 \leq i < j \leq n$ such that $b_i = a_j$, $b_j = a_i$ with $b_i > b_j$, and for all $k \notin \{i, j\}$, $b_k = a_k$.*

The covering relations of the order are analyzed in detail in [4], and the following two lemmas are found out to be very useful. In the original form of the first lemma as appeared in [4], there is a gap, which is pointed out to us by Michelle Wachs. For completeness we write the corrected version along with its proof in here.

Lemma 2.11. *Let $x = (a_1, \ldots, a_n)$ and $y = (b_1, \ldots, b_n)$ be elements of R_n. Suppose that $a_k = b_k$ for all $k = \{1, \ldots, \widehat{i}, \ldots, n\}$ and $a_i < b_i$. Then, $\ell(y) = \ell(x) + 1$ if and only if either*

1. *$0 = a_i$, $b_i = 1$ and $a_j = b_j > 0$ for all $j > i$, or*
2. *$0 < a_i$ and $b_i = a_i + 1$, or*
3. *there exists a sequence of indices $1 \leq j_1 < \cdots < j_s < i$ such that the set $\{a_{j_1}, \ldots, a_{j_s}\}$ is equal to $\{a_i + 1, \ldots, a_i + s\}$, and $b_i = a_i + s + 1$.*

Proof. Let $x, y \in R_n$ be two rooks as in hypothesis and suppose that $\ell(y) = \ell(x) + 1$. Then by Theorem 2.10 we know that y covers x. Obviously, either $b_i = a_i + 1$, or $b_i > a_i + 1$. In the former case, we have two possibilities: either $a_i = 0$, or $a_i > 0$. In the former case, it follows from Proposition 2.8 that unless $a_j = b_j > 0$ for all $j > i$, the equality $\ell(y) = \ell(x) + 1$ cannot be true. In the latter case, there is nothing to say. In conclusion, if $b_i = a_i + 1$, then either 1., or 2. holds. If, on the other hand,

$b_i = a_i + d$ with $d > 1$, then more analysis is needed:

$$\begin{aligned}
\ell(y) &= \sum_{j=1}^{n} b_j^* - coinv(y) \\
&= (\sum_{j=1, j\neq i}^{n} a_j^*) + b_i^* - coinv(y) \\
&= (\sum_{j=1, j\neq i}^{n} a_j^*) + a_i + d + n - i - coinv(y) \\
&= (\sum_{j=1}^{n} a_j^*) + d - coinv(y) \\
&= \ell(x) + d + coinv(x) - coinv(y).
\end{aligned}$$

Hence $d + coinv(x) - coinv(y) = 1$, or $coinv(y) - coinv(x) = d - 1$. We inspect the difference $coinv(x) - coinv(y)$ more closely. If (k, i) with $k < i$ is a coinversion for x, then it stays to be a coinversion for y, as well. Clearly this is also true for the pairs of the form (k, l) where $k < i < l$, or $i < k < l$, or $k < l < i$. Therefore, the difference between $coinv(y)$ and $coinv(x)$ occurs at the pairs of the form

1. (k, i), $k < i$ such that $a_i < a_k < b_i$, or
2. (i, l), $i < l$, such that $a_i < a_l < b_i$.

In the first case, some new coinversions are added, and in the second case some coinversions are deleted. Let us call the number of pairs of the first type by n_1 and the number of pairs of the second type by n_2. Then, $coinv(y) = coinv(x) + n_1 - n_2$, or $coinv(y) - coinv(x) = n_1 - n_2$. Obviously $0 \leq n_1, n_2 \leq d - 1$ (because $b_i = a_i + d$). Hence, we have that $n_1 = d - 1$, and that $n_2 = 0$. Therefore, the following is true: any a_k between a_i and $a_i + d = b_i$ appears before the i-th position. This completes the proof of "if" direction of our lemma.

Next we prove the "only if" direction. If 1. or 2. holds, then $b_i = a_i + 1$, and in this case, it is straightforward to check that $\ell(y) = \ell(x) + 1$. So, we assume that there exists a sequence of indices $1 \leq j_1 < \cdots < j_s < i$ such that the set $\{a_{j_1}, \ldots, a_{j_s}\}$ is equal to $\{a_i + 1, \ldots, a_i + s\}$, and $b_i = a_i + s + 1$. Then

$$\begin{aligned}
\ell(y) &= \sum_{j=1}^{n} b_j^* - coinv(y) \\
&= (\sum_{j=1, j\neq i}^{n} a_j^*) + b_i^* - coinv(y) \\
&= (\sum_{j=1, j\neq i}^{n} a_j^*) + a_i + s + 1 + n - i - coinv(y) \\
&= (\sum_{j=1}^{n} a_j^*) + s + 1 - coinv(y).
\end{aligned}$$

Now it suffices to show that $coinv(y) = s + coinv(x)$. Observe that, when we replace a_i by b_i, the following set of pairs, which are not coinversion pairs for x,

$$\{(j_k, i)|\ k = 1, \ldots, s\},$$

become coinversion pairs for y. Also, upon replacing the entry a_i by b_i, a coinversion pair of x of the form (l, i) or (i, l) (where $l \neq j_k$) stays to be a coinversion pair for y. Therefore,

$$coinv(y) = s + coinv(x),$$

and hence $\ell(y) = \ell(x) + 1$.

□

Example 2.12. Let $x = (4, 0, 5, 0, 3, 1)$, and let $y = (4, 0, 5, 0, 6, 1)$. Then $\ell(x) = 21$, and $\ell(y) = 22$. If $z = (4, 0, 5, 0, 3, 2)$, then $\ell(z) = 22$.

Lemma 2.13. *Let $x = (a_1, \ldots, a_n)$ and $y = (b_1, \ldots, b_n)$ be two elements of R_n. Suppose that $a_j = b_i$, $a_i = b_j$ and $b_j < b_i$ where $i < j$. Furthermore, suppose that for all $k \in \{1, \ldots \widehat{i}, \ldots, \widehat{j}, \ldots, n\}$, $a_k = b_k$. Then, $\ell(y) = \ell(x) + 1$ if and only if for $s = i + 1, \ldots, j - 1$, either $a_j < a_s$, or $a_s < a_i$.*

Example 2.14. Let $x = (2, 6, 5, 0, 4, 1, 7)$, and let $y = (4, 6, 5, 0, 2, 1, 7)$. Then $\ell(x) = 35$, and $\ell(y) = 36$. Let $z = (7, 6, 5, 0, 4, 1, 2)$. Then $\ell(z) = 42$.

3 An EL-labeling of R_n

We call a covering relation in $(R_n, \leq)$ *type 1*, if it is as in Lemma 2.11, and we call it *type 2*, if it is as in Lemma 2.13. Let Γ denote the total order

$$\Gamma = \{0, 1, \ldots, n\} \times \{1, \ldots, n\} \tag{3.1}$$

with respect to lexicographic ordering. Define

$$F : C(R_n) \longrightarrow \Gamma$$

by

$$F(x, y) = \begin{cases} (a_i, b_i), & \text{if } y \text{ covers } x \text{ by type 1} \\ (a_i, a_j), & \text{if } y \text{ covers } x \text{ by type 2.} \end{cases} \tag{3.2}$$

Remark 3.3.

1. If $F(x, y) = (a, b)$ for some $(x, y) \in C(R_n)$, then b is never 0, so Γ is well chosen.

2. If y covers x by type 2, then the set of nonzero entries of y is the same as the set of nonzero entries of x. If y covers x by type 1, then the symmetric difference of the set of nonzero entries of y and the set of nonzero entries of x has at most 2, and at least 1 elements.

Theorem 3.4. *Let $\Gamma = \{0, 1, \ldots, n\} \times \{0, 1, \ldots, n\}$, and let $F : C(R_n) \longrightarrow \Gamma$ be the edge-labeling, defined as in (3.2). Then F is an EL-labeling for R_n.*

We prove our theorem in the next section. Complete labeling of R_3 is shown in Figure 3.1.

4 Proofs

Let Γ be the total order as in (3.1). Then for any $k > 0$, $\Gamma^k = \Gamma \times \cdots \times \Gamma$ is totally ordered with respect to the lexicographic ordering, also. Let $[x, y] \subseteq R_n$ be an interval, and $\mathfrak{c} : x = x_0 < \cdots < x_k = y$ be a maximal chain in $[x, y]$. We define $F(\mathfrak{c})$, the Jordan-Hölder sequence of F-labels of $\mathfrak{c}$ by $F(\mathfrak{c}) = (F(x_0, x_1), \ldots, F(x_{k-1}, x_k)) \in \Gamma^k$, where F is as in (3.2).

Lemma 4.1. *Let $\mathfrak{c} : x = x_0 < x_1 < x_2 = y$ be a maximal chain of $[x, y]$, an interval of length two in R_n. If the Jordan-Hölder sequence of $\mathfrak{c}$ is lexicographically smallest among all Jordan-Hölder sequences from $[x, y]$, then*

$$F(x_0, x_1) \leq F(x_1, x_2). \tag{4.2}$$

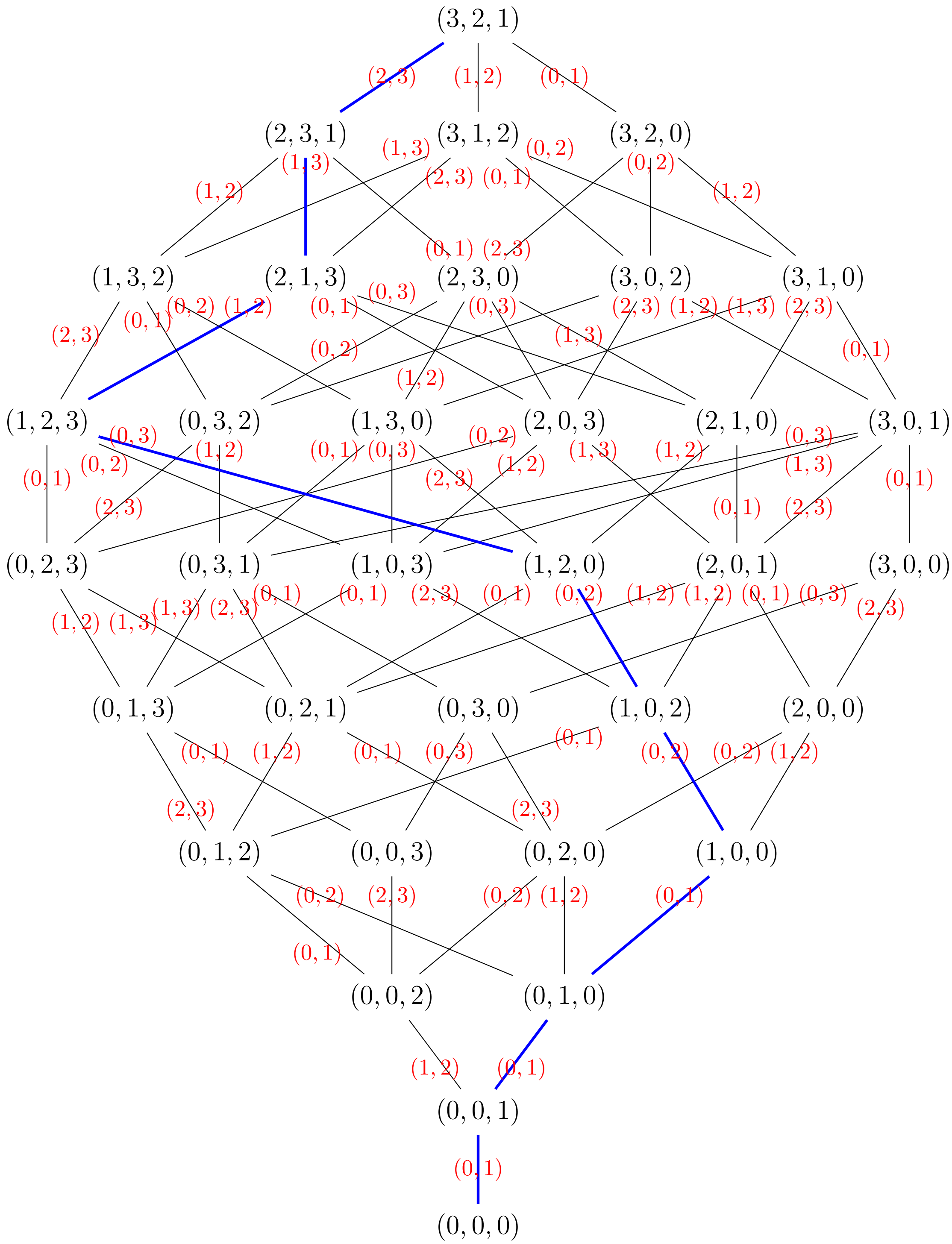

Figure 3.1: EL-labeling of the rook monoid R_3.

Before we start our proof, let us give an example in the case of $n = 3$.

Example 4.3. Let $x = (0, 1, 0)$ and $y = (3, 1, 2)$ in R_3. It is easy to check from Figure 3.1 that in $[x, y]$ the maximal chain

$$\mathfrak{c} : \ x < (1, 0, 0) < (1, 0, 2) < (1, 2, 0) < (1, 2, 3) < (2, 1, 3) < y$$

has the (lexicographically) smallest Jordan-Hölder sequence. Obviously,

$$F(\mathfrak{c}) = ((0, 1), (0, 2), (0, 2), (0, 3), (1, 2), (2, 3))$$

is a non-decreasing sequence.

Proof. Towards a contradiction, assume that (4.2) is not true. Hence,

$$F(x_0, x_1) > F(x_1, x_2). \tag{4.4}$$

Obviously, we have the following cases:

Case 1: $\mathit{type}(x_1, x_2) = 1$, and $\mathit{type}(x_0, x_1) = 1$.
Case 2: $\mathit{type}(x_1, x_2) = 1$, and $\mathit{type}(x_0, x_1) = 2$.
Case 3: $\mathit{type}(x_1, x_2) = 2$, and $\mathit{type}(x_0, x_1) = 1$.
Case 4: $\mathit{type}(x_1, x_2) = 2$, and $\mathit{type}(x_0, x_1) = 2$.

In each case we construct an element $z \in [x, y]$ covering $x_0 = x$ such that $F(x_0, z) < F(x_0, x_1)$. Since by our assumption $F(\mathfrak{c})$ is the lexicographically first Jordan-Hölder sequence, the existence of such an element will give us the contradiction we seek. To this end, let $x_0 = (a_1, \ldots, a_n)$, $x_1 = (b_1, \ldots, b_n)$ and $x_2 = (c_1, \ldots, c_n)$.

Case 1: Since $\mathit{type}(x_0, x_1) = 1$, there exists an index $1 \leq r \leq n$ such that $b_k = a_k$ for all $k \neq r$ and $a_r < b_r$. Likewise, there exists $1 \leq s \leq n$ such that $c_k = b_k$ for all $k \neq s$, and $b_s < c_s$. Thus, $F(x_0, x_1) = (a_r, b_r) > F(x_1, x_2) = (b_s, c_s)$ by our assumption (4.4). Since $a_r < b_r$, we have $r \neq s$, and hence, $b_s = a_s$.

Suppose first that $r > s$. In other words, $a_r < b_r = c_r$ and $a_s = b_s < c_s$ and

$$x_2 = (a_1, \ldots, a_{s-1}, c_s, a_{s+1}, \ldots, a_{r-1}, b_r, a_{r+1}, \ldots, a_n). \tag{4.5}$$

1) If $a_r = b_s = 0$ and $b_r > c_s > 0$, then by condition 1 of Lemma 2.11, we must have $b_r = 1$ and $c_s = 1$, which is impossible.

2) $a_r > b_s = 0$. Then $a_s = 0$. We look at the cases $c_s = 1$ and $c_s > 1$ separately. In the former case, if $a_r = 1$, then define $z = (d_1, \ldots, d_n)$ by setting $d_i = a_i$ for $i \notin \{r, s\}$, and $d_r = 0$, $d_s = 1$. In other words,

$$\begin{aligned}
x_0 &= (a_1, \ldots, a_{s-1}, 0, a_{s+1}, \ldots, a_{r-1}, 1, a_{r+1}, \ldots, a_n),\\
z &= (a_1, \ldots, a_{s-1}, 1, a_{s+1}, \ldots, a_{r-1}, 0, a_{r+1}, \ldots, a_n),\\
x_2 &= (a_1, \ldots, a_{s-1}, 1, a_{s+1}, \ldots, a_{r-1}, b_r, a_{r+1}, \ldots, a_n).
\end{aligned}$$

Then $x_0 < z < x_2$, and $F(x_0, z) = (0, 1) < F(x_0, x_1) = (a_r, b_r) = (1, b_r)$, which is a contradiction. If $a_r > 1$, then define $z = (d_1, \ldots, d_n)$ by setting $d_i = a_i$ for $i \neq s$, and $d_s = 1$. In other words,

$$\begin{aligned}
x_0 &= (a_1, \ldots, a_{s-1}, 0, a_{s+1}, \ldots, a_{r-1}, a_r, a_{r+1}, \ldots, a_n),\\
z &= (a_1, \ldots, a_{s-1}, 1, a_{s+1}, \ldots, a_{r-1}, a_r, a_{r+1}, \ldots, a_n),\\
x_2 &= (a_1, \ldots, a_{s-1}, 1, a_{s+1}, \ldots, a_{r-1}, b_r, a_{r+1}, \ldots, a_n).
\end{aligned}$$

Then $x_0 < z < x_2$, and $F(x_0, z) = (0, 1) < F(x_0, x_1) = (a_r, b_r) = (1, b_r)$, which is a contradiction.

If $c_s > 1$, then possible scenarios are, once more, $a_r = 1$, or $a_r > 1$. In the former case we define z as in

$$\begin{aligned}
x_0 &= (a_1, \ldots, a_{s-1}, 0, a_{s+1}, \ldots, a_{r-1}, 1, a_{r+1}, \ldots, a_n),\\
z &= (a_1, \ldots, a_{s-1}, c_s, a_{s+1}, \ldots, a_{r-1}, 1, a_{r+1}, \ldots, a_n),\\
x_2 &= (a_1, \ldots, a_{s-1}, c_s, a_{s+1}, \ldots, a_{r-1}, b_r, a_{r+1}, \ldots, a_n).
\end{aligned}$$

Then $x_0 < z < x_2$, and $F(x_0, z) = (0, c_s) < F(x_0, x_1) = (a_r, b_r) = (1, b_r)$, which is a contradiction. Finally, we have the case $a_r > 1$ when $c_s > 1$. We have two sub cases; $a_r = c_r$, or $a_r \neq c_r$. In the first sub-case define z as in

$$\begin{aligned}
x_0 &= (a_1, \ldots, a_{s-1}, 0, a_{s+1}, \ldots, a_{r-1}, a_r, a_{r+1}, \ldots, a_n),\\
z &= (a_1, \ldots, a_{s-1}, a_r, a_{s+1}, \ldots, a_{r-1}, 0, a_{r+1}, \ldots, a_n),\\
x_2 &= (a_1, \ldots, a_{s-1}, a_r, a_{s+1}, \ldots, a_{r-1}, b_r, a_{r+1}, \ldots, a_n).
\end{aligned}$$

Then $x_0 < z < x_2$, and $F(x_0, z) = (0, a_r) < F(x_0, x_1) = (a_r, b_r)$, which is a contradiction. In the latter sub-case, since $c_s \neq a_i$ for all $i \in \{1, \ldots, n\}$, we may define z as in

$$x_0 = (a_1, \dots, a_{s-1}, 0, a_{s+1}, \dots, a_{r-1}, a_r, a_{r+1}, \dots, a_n),$$
$$z = (a_1, \dots, a_{s-1}, c_s, a_{s+1}, \dots, a_{r-1}, a_r, a_{r+1}, \dots, a_n),$$
$$x_2 = (a_1, \dots, a_{s-1}, c_s, a_{s+1}, \dots, a_{r-1}, b_r, a_{r+1}, \dots, a_n).$$

Then $x_0 < z < x_2$, and $F(x_0, z) = (0, c_s) < F(x_0, x_1) = (a_r, b_r)$, which is a contradiction.

3) $a_r > b_s > 0$, then we set $d_r = a_s$, $d_s = a_r$ and let $d_k = a_k$ for $k \notin \{s, r\}$. It is clear from Theorem 2.10 that $z = (d_1, \dots, d_n) > x_0$, and from (4.5) that $z < x_2$. Since $F(x_0, z) = (a_s, a_r) < F(x_0, x_1) = (a_r, b_r)$, we have a contradiction.

Next, suppose that $r < s$. Then

$$x_2 = (a_1, \dots, a_{r-1}, b_r, a_{r+1}, \dots, a_{s-1}, c_s, a_{s+1}, \dots, a_n). \tag{4.6}$$

1) $a_r = b_s = 0$, and $b_r > c_s > 0$. Then then by condition 1 of Lemma 2.11, we must have $b_r = 1$ and $c_s = 1$, which is impossible.

2) $a_r > b_s = 0$. Then $a_s = 0$ and by condition 1 of Lemma 2.11, $c_s = 1$. If $a_r = 1$, then define $z = (d_1, \dots, d_n)$ by setting $d_i = a_i$ for $i \neq s$, and $d_s = b_r$. If $a_r > 1$, then define $z = (d_1, \dots, d_n)$ by setting $d_i = a_i$ for $i \neq s$, and $d_s = 1$. Since x_2 is as in (4.5), we see in both cases that $x_0 < z < x_2$, and that $F(x_0, z) = (0, *) < F(x_0, x_1) = (a_r, b_r)$ (for some positive number $*$), which is a contradiction.

3) $a_r > b_s > 0$. If $c_s \neq a_r$, then define $z = (d_1, \dots, d_n)$ by setting $d_k = a_k$ for $k \neq s$, and $d_s = c_s$. Since x_2 is as in 4.6, by using Lemma 2.11 we see that $x_0 < z < x_2$. Since $F(x_0, z) = (a_s, c_r) < (a_r, b_r) = F(x_0, x_1)$, we get a contradiction. If $c_s = a_r$, then define $z = (d_1, \dots, d_n)$ by setting $d_k = a_k$ for $k \neq s$, and $d_s = b_r$. By Lemma 2.11 and Theorem 2.10 $x_0 < z < x_2$. Since $F(x_0, z) = (a_s, b_r) < (a_r, b_r) = F(x_0, x_1)$, we get a contradiction.

Case 2: Since *type*$(x_1, x_2) = 1$, there exists $r \in [n]$ such that $b_k = c_k$ for $k \neq r$, and $b_k < c_k$, and since *type*$(x_0, x_1) = 2$ there exist $i < j$ such that $b_k = a_k$ for $k \notin \{i, j\}$, and $b_i = a_j$, $b_j = a_i$ with $a_i < a_j$. Then $(a_i, a_j) > (b_r, c_r)$ by (4.4). We analyze this case with respect to r in relation with i and j.

1) Suppose first that $r \in \{j+1, \dots, n\}$. Then

$$x_2 = (a_1, \dots, a_{i-1}, a_j, a_{i+1}, \dots, a_{j-1}, a_i, a_{j+1}, \dots, a_{r-1}, c_s, a_{r+1}, \dots, a_n). \tag{4.7}$$

For $k \neq r$ set $d_k = a_k$, and set also $d_r = c_r$. Define $z = (d_1, \dots, d_n)$. Then, $x_0 < z < x_2$ and $F(x_0, z) = (a_r, c_r) < F(x_0, x_1) = (a_i, a_j)$, which is a contradiction.

2) Suppose next $r \in \{1, \dots, i-1\}$. This case is identical to the previous so we skip its proof.

3) Suppose $i < r < j$. Since *type*$(x_0, x_1) = 2$, either $a_r > a_j$, or $a_r < a_i$. If $a_r > a_j$, then $a_r > a_i$. This contradicts with $F(x_1, x_2) = (b_r, c_r) = (a_r, c_r) < F(x_0, x_1) = (a_i, a_j)$. Therefore, $a_r < a_i$. Since *type*$(x_1, x_2) = 1$ and since a_i is at the j-th position, we cannot have $a_i < c_r$. In other words, $a_r < c_r < a_i < a_j$. Now we define $z = (d_1, \dots, d_n)$ by letting $d_k = a_k$ for $k \neq r$, and $d_r = c_r$. It is clear that $x_0 < z < x_2$, and that $F(x_0, z) = (a_r, c_r) < F(x_0, x_1) = (a_i, a_j)$. This is a contradiction, as before.

4) The remaining cases are $r = i$ and $r = j$. If $r = i$, then by our assumption (4.4) we must have $F(x_0, x_1) = (a_i, a_j) > F(x_1, x_2) = (a_j, c_i)$, which is impossible. If $r = j$, then by (4.4) we must have $F(x_0, x_1) = (a_i, a_j) > F(x_1, x_2) = (a_i, c_j)$. Therefore $c_j < a_j$. Define $z = (d_1, \dots, d_n)$ by setting $d_k = a_k$ for $k \neq i$, and $d_i = c_j$. Then $x_0 < z < x_2$ and $F(x_0, z) = (a_i, c_j) < F(x_0, x_1) = (a_i, a_j)$, which is the desired contradiction.

Case 3: Since *type*$(x_0, x_1) = 1$, there exists $r \in [n]$ such that $b_k = a_k$ for $k \neq r$, and $a_r < b_r$, and since *type*$(x_1, x_2) = 2$ there exist $i < j$ such that $b_k = c_k$ for $k \notin \{i, j\}$, and $c_i = b_j$, $c_j = b_i$ with $b_i < b_j$. Then $(b_i, b_j) > (a_r, b_r)$ by (4.4). We analyze this case with respect to r in relation with i and j.

1) Suppose first that $r \in \{j+1, \dots, n\}$. Then

$$x_2 = (a_1, \dots, a_{i-1}, a_j, a_{i+1}, \dots, a_{j-1}, a_i, a_{j+1}, \dots, a_{r-1}, b_s, a_{r+1}, \dots, a_n). \tag{4.8}$$

For $k \notin \{i, j\}$ set $d_k = a_k$, and set also $d_i = a_j$ and $d_j = a_i$. Define $z = (d_1, \dots, d_n)$. Then, $x_0 < z < x_2$ and $F(x_0, z) = (a_i, a_j) < F(x_0, x_1) = (a_r, b_r)$, which is a contradiction.

The case when $r \in \{1, \dots, i-1\}$ is identical, so we skip its proof.

3) Suppose $i < r < j$. Since $type(x_0, x_1) = 1$, and $type(x_1, x_2) = 2$, either $b_r > a_j$, or $b_r < a_i$. The latter case is not possible by our assumption (4.4) which requires $(a_r, b_r) > (a_i, a_j)$. If $b_r > a_j$, then we have two subcases: either $a_r < a_i$, or $a_j < a_r$. Once again, the former case contradicts our assumption (4.4), so it is impossible. In latter case we define $z = (d_1, \dots, d_n)$ by letting $d_k = a_k$ for $k \notin \{i, j\}$, and $d_i = a_j$, $d_j = a_i$. Then $x_0 < z < x_2$, and that $F(x_0, z) = (a_i, a_j) < F(x_0, x_1) = (a_r, b_r)$. A contradiction as before.

4) The remaining cases are $r = i$ and $r = j$. By our assumption (4.4) $F(x_0, x_1) = (a_r, b_r) > F(x_1, x_2) = (b_i, b_j)$. It is clear now that $r = i$ is impossible. If $r = j$, then $b_r = b_j > b_i = a_i$, which forces $a_j = a_r > a_i$ by (4.4). We define $z = (d_1, \dots, d_n)$ by setting $z_k = a_k$ for $k \notin \{i, j\}$, and $z_i = a_j$, $z_j = a_i$. Then $x_0 < z < x_2$ and $F(x_0, z) = (a_i, a_j) < (a_j, b_j) = F(x_0, x_1)$, which is the contradiction we seek.

Case 4: Since $type(x_0, x_1) = 2$, there exist $1 \leq i < j \leq n$ such that $a_i < a_j$, $b_i = a_j$, $b_j = a_i$, and since $type(x_1, x_2) = 2$, there exist $1 \leq r < s \leq n$ such that $b_r < b_s$, $c_r = b_s$ and $c_s = b_r$. If $j < r$ or $s < i$, define $z = (d_1, \dots, d_n)$ by $d_k = a_k$ for $k \notin \{r, s\}$, and $d_s = a_r$, $d_r = a_s$. Then $x_0 < z < x_2$ and $F(x_0, z) = (a_r, a_s) < F(x_0, x_1) = (a_i, a_j)$, which is the contradiction we seek. Therefore, one of the following two cases has to hold:

(a) $i \leq r \leq j \leq s$, or

(b) $r \leq i \leq s \leq j$.

The ways of obtaining desired contradictions in the case of (b) are identical to those of (a), so it is enough to handle the case (a), only. There are several sub cases. Obviously, $i = r$ and $j = s$ at the same time is not possible. The case of $i = r = j = s$ does not make sense. Thus we begin with assuming $j < s$.

1) If $i < r < j < s$, it follows from (4.4) that $a_i > a_r$. Then since $type(x_1, x_2) = 2$, we have $a_j > a_i > a_s$, also. Define $z = (d_1, \dots, d_n)$ by setting $d_k = a_k$ for $k \notin \{r, s\}$ and $d_r = a_s$, $d_s = a_r$. Then $x_0 < z < x_2$ and $F(x_0, z) = (a_r, a_s) < (a_i, a_j) = F(x_0, x_1)$, which is the contradiction we seek.

2) If $i = r < j < s$, then $b_s > b_r = b_i = a_j > a_i$. This contradicts with (4.4), which requires $(a_i, a_j) > (b_r, b_s)$.

3) If $i < r = j < s$, then the inequalities $a_i < a_j = a_r$ and $a_i = b_r < b_s = a_s$ must hold. By (4.4), $F(x_0, x_1) = (a_i, a_r) > F(x_1, x_2) = (a_i, a_s)$. Thus $a_r > a_s$. We define $z = (d_1, \ldots, d_n)$ by setting $d_k = a_k$ for $k \notin \{i, s\}$, and $d_i = a_s$, $d_s = a_i$. Note that, since $type(x_0, x_1) = 2$, by Lemma 2.13, for $i < l < r$ either $(d_l =)a_l < a_i$, or $(d_l =)a_l > a_j = a_r > a_s$ holds. We already know $d_r = a_r > a_s$. Note also that, since $type(x_1, x_2) = 2$, for $r < l < s$ we have either $(d_l = a_l =)b_l < b_r = a_i$, or $(d_l = a_l =)b_l > b_s = a_s$. In conclusion, for $i < l < s$ we have either $d_l < a_i$, or $d_l > a_s$. In other words, x_0 is covered by z. Since

$$z = (a_1, \ldots, a_{i-1}, a_j = a_r, a_{i+1}, \ldots, a_{j-1}, a_s, a_{j+1}, \ldots, a_{s-1}, a_i, a_{s+1}, \ldots, a_n),$$

we see also that $z < x_2$. Since $F(x_0, z) = (a_i, a_s) < F(x_0, x_1) = (a_i, a_j)$ we obtain the desired contradiction.

Next case is when (a) holds with $j = s$. The only remaining possibility, in this case, is that $i < r < j = s$. Then $a_i < a_j$ and $a_r = b_r < b_s = a_i$. Define $z = (d_1, \ldots, d_n)$ by setting $d_k = a_k$ for $k \notin \{r, j\}$ and $d_j = a_r$, $d_r = a_j$. It is clear that $x_0 < z$. Since $type(x_0, x_1) = 2$, by Lemma 2.13, for any $i < l < j = s$ we have either $d_l < a_i$, or $d_l > a_j$. In other words, if $d_l > a_i$, then it must be true that $d_l > a_j$. On the other hand, since $type(x_1, x_2) = 2$, we know that $r < l < j = s$ $(d_l = a_l) = b_l > a_i$, or $(d_l = a_l =)b_l < a_r$. In conclusion, for $r < l < j$ we have $d_l > a_j$, or $d_l < a_r$. Hence, x_2 covers z (by Lemma 2.13). Since $F(x_0, z) = (a_r, a_i) > F(x_0, x_1) = (a_i, a_j)$, we obtain our desired contradiction.

The proof is complete and the chain which is lexicographically first in an interval of length 2 has to be increasing. □

Proposition 4.9. *Let $[x, y]$ denote an interval in R_n and suppose $\mathfrak{c} : x = x_0 < \cdots < x_k = y$ is a maximal chain whose Jordan-Hölder sequence $F(\mathfrak{c})$ is lexicographically smallest among all Jordan-Hölder sequences for $[x, y]$. Then,*

$$F(x_0, x_1) \leq F(x_1, x_2) \leq \cdots \leq F(x_{k-1}, x_k).$$

Proof. Assume otherwise that there exists $t \in [k-1]$ such that $F(x_{t-1}, x_t) > F(x_t, x_{t+1})$. Then it follows from Lemma 4.1 that $x_{t-1} < x_t < x_{t+1}$ cannot be the lexicographically smallest chain in $[x_{t-1}, x_{t+1}]$, which implies that the chain $\mathfrak{c}$ cannot be lexicographically smallest. □

Proposition 4.10. *We use the notation of Proposition 4.9. There exists a unique maximal chain $x = x_0 < \cdots < x_k = y$ with $F(x_0, x_1) \leq \cdots \leq F(x_{k-1}, x_k)$.*

Proof. We already know that the lexicographically first chain is increasing. Therefore, it is enough to show that there is no other increasing chain. We prove this by induction on the length of the interval $[x, y]$. Clearly, if y covers x, there is nothing to prove. So, we assume that in an interval of length $k(\geq 2)$ there exists a unique increasing maximal chain, and let $[x, y]$ be an interval of length $k + 1$. Suppose

$$\mathfrak{c} : x = x_0 < x_1 < \cdots < x_k < x_{k+1} = y$$

is the maximal chain that is lexicographically first in Γ^{k+1}. Towards a contradiction assume that there exists another increasing chain

$$\mathfrak{c}' : x = x_0 < x_1' < \cdots < x_k' < x_{k+1} = y.$$

Since the length of the chain

$$x_1' < \cdots < x_k' < x_{k+1} = y$$

is k, by our induction hypotheses it is the lexicographically first chain between x_1' and y. Now it is enough to refute each of the following possibilities to conclude that the maximal chain $\mathfrak{c}'$ does not increase, hence $\mathfrak{c}$ is unique.

Case 1: $\mathit{type}(x_0, x_1) = 1$, and $\mathit{type}(x_0, x_1') = 1$,
Case 2: $\mathit{type}(x_0, x_1) = 1$, and $\mathit{type}(x_0, x_1') = 2$,
Case 3: $\mathit{type}(x_0, x_1) = 2$, and $\mathit{type}(x_0, x_1') = 1$,
Case 4: $\mathit{type}(x_0, x_1) = 2$, and $\mathit{type}(x_0, x_1') = 2$.

Suppose $x_0 = (a_1, \ldots, a_n)$, $x_1 = (b_1, \ldots, b_n)$, and $x_1' = (c_1, \ldots, c_n)$.

Case 3: Suppose that x_1 covers x_0 by interchanging a_i and a_j (where $i < j$), and that x_1' covers x_0 by the type 1; replacing a_r with c_r. Since $(a_i, a_j) = F(x_0, x_1) \leq F(x_0, x_1') = (a_r, c_r)$, $a_i \leq a_r < c_r$. In fact, $(a_i, a_j) < (a_r, c_r)$.

Assume first that $r < i$. Define $z = (e_1, \ldots, e_n) \in R_n$ by $e_k = a_k$ for $k \notin \{r, i, j\}$ and $e_r = c_r$, $e_i = a_j$ and $e_j = a_i$. It is easy to check that z covers x_1', and $F(x_1', z) = (a_i, a_j)$. Since the Jordan-Hölder sequence of $x_1' < \cdots < x_n' < x_{n+1} = y$ is lexicographically smallest in $[x_1', y]$, and since $F(\mathfrak{c}')$ is increasing,

$$(a_r, c_r) = F(x, x_1') \leq F(x_1', x_2') \leq F(x_1', z) = (a_i, a_j).$$

This contradicts with $(a_i, a_j) < (a_r, c_r)$. Therefore, we may assume that $r \geq i$. A similar argument shows that we may assume $r \leq j$ also.

Assume that $r = i$. Since $type(x_0, x'_1) = 1$, any number between a_i and c_i has to occur before the i-th position. This contradicts with $(a_i, a_j) < (a_r, c_r) = (a_i, c_r)$.

Next, assume that $r = j$. Since $type(x_0, x'_1) = 1$, any number between a_j and c_j has to occur before the j-th position. If all of them occur before i-th position, we define $z = (e_1, \ldots, e_n)$ by $e_k = a_k$ for $k \notin \{i, j\}$ and $e_i = c_j$, $e_j = a_i$. Then, z covers x'_1 and $F(x'_1, z) = (a_i, c_j)$. This contradicts with

$$(a_j, c_j) = F(x, x'_1) \leq F(x'_1, x'_2) \leq F(x'_1, z) = (a_i, c_j).$$

If any of the numbers between a_j and c_j occur between the i-th and the j-th positions, define $z = (e_1, \ldots, e_n)$ as follows. Let $i < m < j$ be the smallest number such that $a_j < a_m < c_j$. Let $e_k = a_k$ for $k \notin \{i, m, j\}$, and let $e_i = a_m,\ e_m = a_i,\ e_j = c_j$. Then, z covers x'_1, and $F(x'_1, z) = (a_i, a_m)$. Since $(a_j, c_j) = F(x, x'_1)$, we have a contradiction in this case also.

Finally, assume that $i < r < j$. Define $z = (e_1, \ldots, e_n)$ by $e_k = a_k$ for $k \notin \{i, r, j\}$, and $e_i = a_j$, $e_j = a_i$, $e_r = c_r$. It is easy to check that z covers x'_1, and that $F(x'_1, z) = (a_i, a_j)$. Since $(a_j, c_j) = F(x_0, x'_1)$, we obtain a contradiction here as well. This completes Case 3.

Case 4: Suppose that x_1 covers x_0 by interchanging a_i and a_j (where $i < j$), and that x'_1 covers x_0 by interchanging a_r and a_s (where $r < s$). In the following situations "$r < s < i < j$, $i < j < r < s$, $r < i < j < s$, $i < r < s < j$, $i < r < j < s$, $r < i = s < j$, $r < i < s = j$," define $z = (e_1, \ldots, e_n)$ by $e_k = c_k$ for $k \notin \{i, j\}$, and $e_i = a_j = b_i$, $e_j = a_i = b_j$. Then, z covers x'_1, and $F(x'_1, z) = (a_i, a_j)$. This contradicts with

$$(a_r, a_s) = F(x, x'_1) \leq F(x'_1, x'_2) \leq F(x'_1, z) = (a_i, a_j).$$

The remaining cases are $r \leq i \leq s \leq j$, and $i \leq r \leq j \leq s$. If $r < i < s < j$, define $z = (e_1, \ldots, e_n)$ by $e_k = b_k$ for $k \notin \{i, s\}$, and $e_i = b_s = a_r$, $e_s = b_i = a_i$. Then, $F(x'_1, z) = (a_i, a_r)$ gives the contradiction.

If $r = i < s < j$, define $z = (e_1, \ldots, e_n)$ by $e_k = b_k$ for $k \notin \{s, j\}$, and let $e_s = b_j = a_j$, $e_j = b_s = a_i$. Then, $F(x'_1, z) = (a_i, a_j)$ gives the contradiction.

If $r = i < j < s$, since $type(x_0, x'_1) = 2$, we see that $a_j > a_s$. This contradicts $F(x_0, x_1) = (a_i, a_j) < (a_i, a_s) = F(x_0, x'_1)$.

Next, assume that $i < r = j < s$. Assume also that there exists an index $i < m < j$ such that $a_j < c_m = a_m < a_s$. Let m' be the smallest such index. Define $z = (e_1, \ldots, e_n)$ by $e_k = c_k$ for $k \notin \{i, m'\}$, and $e_i = c_{m'} = a_{m'}$, $e_{m'} = c_i = a_i$. Then, $F(x_1', z) = (a_i, a_{m'}) < (a_j, a_s) = F(x_0, x_1')$. This gives a contradiction as before. Therefore, we may assume that there does not exist any $i < m < j$ such that $a_j < a_m < a_s$. Then, for any $i < m < j$ we have either $c_m = a_m < c_i = a_i$, or $c_m = a_m > c_j = a_s$. In this case, define $z = (e_1, \ldots, e_n)$ by $e_k = c_k$ for $k \notin \{i, m'\}$, and $e_i = c_{m'} = a_{m'}$, $e_{m'} = c_i = a_i$. Then, $F(x_1', z) = (a_i, a_{m'}) < (a_j, a_s) = F(x_0, x_1')$ provides a contradiction, as before.

The final case is $i < r < j = s$. Observe that $a_r < a_i$ is forced. Thus, $F(x_0, x_1) = (a_i, a_j) < (a_r, a_s) = F(x_0, x_1')$ is a contradiction. Notice $a_i = a_r = 0$ is impossible, also. This completes Case 4.

Case 1: There exists $1 \leq i \leq n$ such that $b_k = a_k$ for all $k \neq i$, and $b_i > a_i$, and there exists $1 \leq r \leq n$ such that $c_k = a_k$ for $k \neq r$, and $c_r > a_r$. Note that $r = i$ is impossible. Define $z = (e_1, \ldots, e_n)$ by $e_k = c_k$ for $k \neq i$, and $e_i = b_i$. Then, $F(x_1', z) = (a_i, b_i) < (a_r, c_r) = F(x_0, x_1')$ gives the contradiction.

Case 2: Suppose that x_1 covers x_0 by replacing a_i by b_i, and x_1' covers x_0 by interchanging a_r and a_s, where $r < s$.

If $i \leq r < s$, define $z = (e_1, \ldots, e_n)$ by $e_k = c_k$ for $k \neq i$, and $e_i = b_i$. Then, $F(x_1', z) = (a_i, b_i) < (a_r, a_s) = F(x_0, x_1')$ gives the contradiction.

Assume that $r < i < s$. Observe that a_r cannot be equal to a_i, otherwise, $a_r = a_i = 0$ forcing $type(x_0, x_1') \neq 2$. Therefore, we may assume that $a_i < a_r$. Then, either $b_i < a_r$, or $a_i < a_r < b_i$. If $a_r > b_i$, define $z = (e_1, \ldots, e_n)$ by $e_k = c_k$ for $k \neq i$, and $e_i = b_i$. Then, $F(x_1', z) = (a_i, b_i) < (a_r, a_s) = F(x_0, x_1')$. This gives a contradiction as before, so, we may assume that $a_i < a_r < b_i$. Since $type(x_0, x_1) = 1$, any number between a_i and b_i (hence, any number between a_i and a_r) occurs before the i-th position. Since $type(x_0, x_1') = 2$, we know that $c_s = a_r$ and $c_i = a_i$, and furthermore if $i < k < s$, then either $c_k < a_r = c_s$, or $c_k > a_s = c_r$. Define $z = (e_1, \ldots, e_n)$ by $e_k = c_k$ for $k \notin \{i, s\}$, and $e_i = c_s = a_r$, $e_s = c_i = a_i$. Clearly $x_1' \leq z$. For $i < k < s$, either $e_k < a_i = c_i$, or $e_k > c_s = a_r$. Therefore, z covers x_1' and $F(x_1', z) = (a_i, a_r) < (a_r, a_s) = F(x_0, x_1')$ gives the contradiction.

If $r < s < i$, define $z = (e_1, \dots, e_n)$ by $e_k = c_k$ for $k \neq i$, and $e_i = b_i$. Then, $F(x_1', z) = (a_i, b_i) < (a_r, a_s) = F(x_0, x_1')$ gives the contradiction.

Finally, observe that $r < i = s$ is impossible. Otherwise a_r has to be less than a_i which contradicts the assumption that $F(x_0, x_1) = (a_i, b_i) < (a_r, a_i) = F(x_0, x_1')$. This completes Case 2 as well as the proof of our proposition.

□

Proof of Theorem 3.4. Let $\Gamma = \{0, 1, \dots, n\} \times \{0, 1, \dots, n\}$, and let $F : C(R_n) \longrightarrow \Gamma$ be the edge-labeling, as defined in (3.2). By Propositions 4.9 and 4.10, $F : C(R_n) \longrightarrow \Gamma$ is an EL-labeling . □

It is well known that an interval of length two in a Weyl group consists of four elements. However, for Renner monoids, this is not true. In R_n, already for $n = 2$, there are intervals of length two with three elements. We finish this section by proving that in R_n an interval of length two has at most four elements.

Proposition 4.11. *Let $[x, y] \subseteq R_n$ be an interval of length two. Then, $[x, y]$ is either a chain, or a diamond. In other words, either $[x, y] = \{x, x_1, x_1', y\}$ with $x < x_1 \neq x_1' < y$, or $[x, y] = \{x, x_1, y\}$ with $x < x_1 < y$.*

Proof. Let $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ denote x and y, respectively. Let J denote the set $\{i \in [n] : \ a_i \neq b_i\}$. If $\ell(y) - \ell(x) = 2$, then by Lemmas 2.11 and 2.13, we see that $|J| \leq 4$. Therefore, if $z = (c_1, \dots, c_n) \in [x, y]$ is strictly between x and y, then

$$\{i \in [n] : \ c_i \neq a_i \text{ or } c_i \neq b_i\}$$

has at most four elements, also. Arguing case by case, as in the proof of Proposition 4.10, we see that there are at most two different possibilities for z. □

5 Final remarks

The Möbius function $\mu : I(P) \longrightarrow \mathbb{Z}$ is an integer valued function defined on the set of all intervals of $\hat{P}$, uniquely determined by the following conditions

$$\mu([x, y]) = \begin{cases} 1 & \text{if } x = y, \\ -\sum_{x \leq z < y} \mu([x, z]) & \text{if } x < y. \end{cases}$$

Let $R_{n,k} \subset R_n$, $k = 0, \dots, n$ denote the subposet consisting of elements whose rank is k. In [1] it is shown that the Möbius function on $I(R_{n,k})$ takes values in $\{-1, 0, 1\}$.

When $k = n$, $R_{n,k}$ is the symmetric group, and the Möbius function on S_n is well known. See [18], [17], [8].

There is a remarkable topological interpretation of the Möbius function on $\Delta(P)$. For $x, y \in P$

$$\mu((x,y)) = \widetilde{\chi}(\mathrm{lk}_{x,y}),$$

where $\mathrm{lk}_{x,y}$ is the order complex of the open interval $(x,y) = \{z \in P : \ x < z < y\}$, and $\widetilde{\chi}$ is the reduced Euler characteristic of the topological space $\mathrm{lk}_{x,y}$. In particular, $\mu(\hat{P}) := \mu([\hat{0}, \hat{1}])$ is the reduced Euler characteristic of $\Delta(P)$.

Our next result is the determination of the homoemorphism type of the simplicial complex $\Delta(R_n)$. To this end, we recall the following important result of Danaraj and Klee.

Lemma 5.1 ([5], page 444)**.** *Let Δ denote a pure shellable simplicial complex in which every codimension one face is contained in at most two facets. If every codimension one face is contained in exactly two facets, then Δ is homeomorphic to a sphere. Otherwise, Δ is homeomorphic to a ball.*

Theorem 5.2. *The order complex $\Delta((x,y))$ of every open interval $(x,y) \subset R_n$ has the homotopy type of a sphere or a ball. In particular, $\Delta(R_n)$ triangulates a ball of dimension n^2.*

Proof. It follows from Proposition 4.11 that every codimension one face of an open interval (x,y) in R_n is contained in at most two facets. Thus, by Theorem 3.4 and Lemma 5.1 we see that $\Delta((x,y))$ has the homotopy type of a sphere or a ball.

For our second claim, first observe that the dimension of the dense $B \times B$ orbit in M_n is n^2. Therefore, the rank (as a poset) of R_n is n^2, and hence, $\dim \Delta(R_n) = n^2$. To see that $\Delta(R_n)$ is homeomorphic to a ball, it suffices to show that the reduced Euler characteristic of $\Delta(R_n)$ is 0.

By Proposition 3.8.5 and Theorem 3.14.2 of [16], we know that $(-1)^{n^2-1}\mu(\widehat{R_n})$ is equal to the number of strictly decreasing maximal chains in R_n. (We should note here that, after identifying the (lexicographically) totally ordered set $\{(i,j) : \ 0 \leq i < j \leq n\}$ with $\{1, \ldots, n(n+1)/2\}$, our EL-labeling gives an R-labeling in the sense of [16].) On the other hand, it is easy to check that in our EL-labeling of R_n, there is no maximal chain with strictly decreasing labels. Indeed, the label of the covering relation $(\hat{0}, (0, \ldots, 0, 1))$ is $(0,1)$, and the labels of the succeeding covering relations $((0, \ldots, 0, 1), (0, \ldots, 0, 2))$ and $((0, \ldots, 0, 1), (0, \ldots, 0, 1, 0))$ are $(1,2)$ and $(0,1)$, respectively. (See Figure 3.1 for $n = 3$ case). This shows that $(-1)^{n^2-1}\mu(\widehat{R_n}) = 0$, hence $\widetilde{\chi}(\Delta(R_n)) = 0$

□

References

[1] Kürşat Aker, Mahir Bilen Can, and Müge Taşkin. R-polynomials of finite monoids of Lie type. *Internat. J. Algebra Comput.*, 20(6):793–805, 2010.

[2] Anders Björner. Shellable and Cohen-Macaulay partially ordered sets. *Trans. Amer. Math. Soc.*, 260(1):159–183, 1980.

[3] Anders Björner and Michelle Wachs. Bruhat order of Coxeter groups and shellability. *Adv. in Math.*, 43(1):87–100, 1982.

[4] M.B. Can and L.E. Renner. Bruhat-chevalley order on the rook monoid. *Turkish J. Math.*, 35(2):1–21, 2011.

[5] Gopal Danaraj and Victor Klee. Shellings of spheres and polytopes. *Duke Math. J.*, 41:443–451, 1974.

[6] M. J. Dyer. Hecke algebras and shellings of Bruhat intervals. *Compositio Math.*, 89(1):91–115, 1993.

[7] Paul H. Edelman. The Bruhat order of the symmetric group is lexicographically shellable. *Proc. Amer. Math. Soc.*, 82(3):355–358, 1981.

[8] Brant C. Jones. An explicit derivation of the Möbius function for Bruhat order. *Order*, 26(4):319–330, 2009.

[9] Edwin A. Pennell, Mohan S. Putcha, and Lex E. Renner. Analogue of the Bruhat-Chevalley order for reductive monoids. *J. Algebra*, 196(2):339–368, 1997.

[10] Robert A. Proctor. Classical Bruhat orders and lexicographic shellability. *J. Algebra*, 77(1):104–126, 1982.

[11] Mohan S. Putcha. *Linear algebraic monoids*, volume 133 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1988.

[12] Mohan S. Putcha. Shellability in reductive monoids. *Trans. Amer. Math. Soc.*, 354(1):413–426 (electronic), 2002.

[13] Lex E. Renner. Analogue of the Bruhat decomposition for algebraic monoids. *J. Algebra*, 101(2):303–338, 1986.

[14] Lex E. Renner. *Linear algebraic monoids*, volume 134 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2005. Invariant Theory and Algebraic Transformation Groups, V.

[15] Louis Solomon. An introduction to reductive monoids. In *Semigroups, formal languages and groups (York, 1993)*, volume 466 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 295–352. Kluwer Acad. Publ., Dordrecht, 1995.

[16] Richard P. Stanley. *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012.

[17] John R. Stembridge. A short derivation of the Möbius function for the Bruhat order. *J. Algebraic Combin.*, 25(2):141–148, 2007.

[18] Daya-Nand Verma. Möbius inversion for the Bruhat ordering on a Weyl group. *Ann. Sci. École Norm. Sup. (4)*, 4:393–398, 1971.

[19] Michelle L. Wachs. Poset topology: tools and applications. In *Geometric combinatorics*, volume 13 of *IAS/Park City Math. Ser.*, pages 497–615. Amer. Math. Soc., Providence, RI, 2007.