Combinatorial models for the variety of complete quadrics

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Abstract

We develop several combinatorial models that are useful in the study of the SL_n-variety $\mathcal{X}$ of complete quadrics. Barred permutations that we introduce parameterize the fixed points of the action of a maximal torus T of SL_n, while μ-involutions parameterize the orbits of a Borel subgroup of SL_n. Using these combinatorial objects, we characterize the T-stable curves and surfaces on $\mathcal{X}$, compute the T-equivariant K-theory of $\mathcal{X}$, and describe a Białynicki-Birula cell decomposition for $\mathcal{X}$. Furthermore, we give a characterization of the Bruhat order on Borel orbits in $\mathcal{X}$.

1 Introduction

The variety of complete quadrics has a rich history in algebraic geometry, dating back to its use by Chasles to determine the number, 3264, of plane conics tangent to five given conics [11]. The variety of complete quadrics also plays a pivotal role in the theory of spherical varieties, where it serves as the motivating example for wonderful spherical varieties, first introduced in [12]. In this paper, we discuss several combinatorial models that are useful in characterizing fundamental geometric properties of the variety of complete quadrics.

Let n be a natural number. The variety $\mathcal{X}$ of compete quadrics in n variables is equipped with an action of the special linear group SL_n. The SL_n-orbits of $\mathcal{X}$ are naturally parameterized by compositions of n. For a fixed composition μ, the SL_n-orbit $\mathscr{O}^\mu$ possesses a finer decomposition into orbits under a Borel subgroup B of SL_n. The B-orbits in $\mathscr{O}^\mu$ are parameterized by combinatorial objects called *μ-involutions*. Roughly, these are permutations of $1, 2, \dots, n$ that when subdivided into strings whose lengths are given by the parts of μ, have each string represent an involution of its alphabet.

A *degenerate involution* of length n is a μ-involution for some composition μ of n. Thus, the set of all B-orbits on $\mathcal{X}$ is parameterized by degenerate involutions of length n. Given a degenerate involution π, let $\mathcal{X}^\pi$ denote the closure of the B-orbit associated to π. Extrapolating from the observation that the Bruhat order on the symmetric group S_n can be identified with the inclusion order on Schubert varieties in the complete flag variety, we define an analogue of Bruhat order on the set of degenerate involutions. Namely, $\pi \leq \pi'$ if and only if $\mathcal{X}^\pi \subseteq \mathcal{X}^{\pi'}$. In Section 5, we summarize what is known about Bruhat order for degenerate involutions.

The second combinatorial object that we introduce is the notion of a *barred permutation*. Barred permutations parameterize the fixed points of a maximal torus $T \subseteq B$ of SL_n. Because each B-orbit has at most one T-fixed point, barred permutations can be thought of as certain degenerate involutions. A degenerate involution is a barred permutation if the associated composition has only 1's and 2's as its parts and if whenever it has a part of length 2, the corresponding string is decreasing, corresponding to the non-trivial involution of the two numbers.

Using these notions, we describe several key geometric features of $\mathcal{X}$. We prove the following characterization of T-stable curves and surfaces on $\mathcal{X}$ (see Section 4).

Theorem 1.1. All T-stable curves on $\mathcal{X}$ are isomorphic to $\mathbb{P}^1$ and all T-stable surfaces on $\mathcal{X}$ are isomorphic to $\mathbb{P}^2$. If a T-stable curve C on $\mathcal{X}$ is not contained on a T-stable surface, then its two T-fixed points correspond to barred permutations (of the same composition type μ) which differ by interchanging two numbers i and j in different strings. The weight of C is the root $\alpha_{ij} := \epsilon_i - \epsilon_j$ in the notation of [4]. A T-stable surface Y on $\mathcal{X}$ has three T-fixed points whose corresponding barred permutations correspond to two composition types μ and μ'. Then μ' is obtained from μ by refinement, replacing a single part 2 with two 1 parts. If ji is the string in that part of the barred permutation of type μ, then the other two barred permutations are identical except that the string ji becomes $i|j$ or $j|i$. The codimension one torus whose kernel is the root α_{ij} fixes each point in Y pointwise.

Using this result and work of Banerjee-Can [1], we are able to give a complete description of the T-equivariant K-theory of $\mathcal{X}$ in Theorem 4.9.

Our combinatorial objects are also used to study a Białynicki-Birula decomposition of $\mathcal{X}$, which decomposes $\mathcal{X}$ into a union of affine cells based on the flow along an admissible one-parameter subgroup toward T-fixed points on $\mathcal{X}$. The structure of Białnicki-Birula cells for spherical varieties is studied in various degrees of generality: for smooth projective spherical varieties, by Brion-Luna [7]; for wonderful compactifications of symmetric varieties, by De Concini-Springer [13] and for the variety $\mathcal{X}$, by Strickland [20]. In the latter work, Strickland enumerates the number of cells of each dimension. A key result of [7] implies that the intersection of a Białynicki-Birula cell on $\mathcal{X}$ with an SL_n-orbit is either empty or equal to a single B-orbit.

We provide two combinatorial maps that describe which B-orbits on $\mathcal{X}$ constitute each Białynicki-Birula cell. The two maps τ, which sends degenerate involutions to barred permutations, and σ, which sends barred permutations to degenerate involutions, have the following geometric interpretations. Given a degenerate involution π, $\tau(\pi)$ is the barred permutation parameterizing the unique T-fixed point in the cell containing the B-orbit of π (Proposition 6.3). Conversely, given a barred permutation β, $\sigma(\beta)$ is the degenerate involution corresponding to the B-orbit which is dense in the cell flowing to the T-fixed point parameterized by β (Propsition 6.5).

We also characterize Bruhat order for degenerate involutions, using ideas of Richardson-Springer [18], Timashev [21] and Brion [6]. The restriction of Bruhat order to μ-involutions for a fixed μ coincides with the opposite of the usual Bruhat order on permutations when $\mu = (1, 1, \dots, 1)$, in which case the degenerate involutions are identified with permutations, and when $\mu = (n)$, in which case the corresponding degenerate involutions are identified with ordinary involutions [19]. However, we show that the same property does not hold for all μ in general. Instead, we give a characterization of Bruhat order covering relations in Theorem 5.11.

We now describe the organization of the paper. In Section 2 we give some background information. Our definition of degenerate involutions is found in Definiton 2.3. Section 3 introduces barred permutations and counts the number of barred permutations of given length. In Section 4 we describe the T-stable curves and surfaces on $\mathcal{X}$, culminating in a description of the T-equivariant and SL_n-equivariant K-theory of $\mathcal{X}$. Then we discuss Bruhat order on degenerate involutions in Section 5. Finally, we study properties of cell decompositions in Section 6.

2 Combinatorial and geometric preliminaries

Let n be a positive number. The ordered set $\{1, \dots, n\}$ is denoted by $[n]$. All algebraic groups and varieties are defined over $\mathbb{C}$ for convenience, but our results can be extended over any algebraically closed field of characteristic $\neq 2$.

All posets are assumed to be finite and assumed to have a maximal and minimal element. Recall that such a poset P is *graded* if every maximal chain in P has the same length. We denote by $\mathrm{rk} : P \to \mathbb{N}$ the *rank function* on P so that for $x \in P$, $\mathrm{rk}(x)$ is the length of a maximal chain from the minimal element to x. The *rank* of a graded poset P, denoted by $\mathrm{rk}(P)$, is defined to be the rank of the maximal element.

When a solvable group B acts on a projective variety with finitely many orbits, the poset consisting of irreducible B-stable subvarieties with respect to inclusion ordering is almost a graded poset [17, Exercise 8.9.12]. The only subtlety is that there may be more than one minimal element. In this paper, the posets we consider always have a unique minimal element, so we do not concern ourselves with this issue.

Throughout this paper, we let G be a reductive algebraic group. We denote by T, B, and W a maximal torus of G, a Borel subgroup of G containing T, and the Weyl group $W := N_G(T)/T$ of G. When $G = SL_n$, the special linear group of $n \times n$ matrices with determinant 1, we take T to be the subgroup of diagonal matrices and B to be the subgroup of upper triangular matrices. In this case, the Weyl group W is identified with S_n, the symmetric group of permutations on n letters. We write the cycle representation of a permutation using parentheses and its one-line notation using brackets. For example, both $(3, 5)$ and $[125436]$ denote the permutation in S_6 that interchanges 3 and 5 while fixing the other four numbers. The length of a permutation $w \in S_n$, denoted $\ell(w)$, is the number of inversions of w, that is to say the number of pairs (i, j) with $1 \leq i < j \leq n$ and $w(i) > w(j)$.

2.1 Bruhat-Chevalley order for involutions

An involution is an element of S_n of order ≤ 2. We denote by $\mathcal{I}_n$ the set of involutions in S_n. The elements of $\mathcal{I}_n$ are in one-to-one correspondence with B-orbits in the quasi-projective variety of non-degenerate quadric hypersurfaces in $\mathbb{A}^n$. Under this correspondence, an involution π with two-cycles $(a_1, b_1), \ldots, (a_k, b_k)$ and one-cycles $c_1, \ldots, c_m$ corresponds to the B-orbit of the distinguished quadric $Q_\pi := x_{a_1}x_{b_1}+\cdots+x_{a_k}x_{b_k}+x_{c_1}^2+\cdots+x_{c_m}^2$. Therefore, similar to the Bruhat ordering on S_n, there is an induced Bruhat order on involutions given by inclusion of B-orbit closures. The Bruhat order on $\mathcal{I}_n$ agrees with the opposite of the restriction of Bruhat order on S_n [19].

The Bruhat order on $\mathcal{I}_n$ is graded, but with a different rank function [15]. Explicitly, for $\pi \in \mathcal{I}_n$, the corank function is given by

$$L(\pi) = L_{(n)}(\pi) := \frac{\ell(\pi) + \mathrm{exc}(\pi)}{2}, \tag{1}$$

where $\mathrm{exc}(w)$, the *exceedance* of $w \in S_n$, is defined by

$$\mathrm{exc}(w) := \#\{i \in [n] : \ w(i) > i\}.$$

Thus, the rank function is given by $\mathrm{rank}(\pi) := L(\min) - L(\pi)$, where min is the involution $k \mapsto n + 1 - k$. Note that the exceedance of an involution is the number of 2-cycles that appear in its cycle decomposition.

2.2 The variety of complete quadrics

A G-variety is said to be spherical if it contains a dense B-orbit. The variety $\mathcal{X}_0$ of non-singular quadric hypersurfaces in $\mathbb{P}^{n-1}$ is a spherical and homogeneous SL_n-variety. Explicitly, by identifying a non-singular quadratic hypersurface Q in n variables with a full rank $n \times n$ symmetric matrix A such that

$$Q(x_1, x_2, \ldots, x_n) = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \end{pmatrix} A \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix},$$

up to scalar multiplication, the action is given by $g \cdot [A] = [gAg^T]$ for $g \in SL_n$, where g^T is the transpose of g. Then the stabilizer of $Q_{\mathrm{id}} = x_1^2 + x_2^2 + \cdots + x_n^2$, which corresponds to $A = I_n$, the $n \times n$ identity matrix, is the special orthogonal group SO_n.

It is natural to study completions of $\mathcal{X}_0$, and while the projectivization of all nonzero $n \times n$ symmetric matrices (adding the singular quadric hypersurfaces to the boundary) is one such completion, there is a more useful completion to consider, namely the *wonderfulembedding* of $\mathcal{X}_0$.

In general, if X is a smooth complete G-variety, and $X_0 \subset X$ is a homogenous G-subvariety which is dense in X, then X is called a wonderful embedding of X_0 if

1. $X \setminus X_0$ is the union of finitely many G-stable smooth codimension one subvarieties X^i for $i = 1, 2, \ldots, r$;

2. for any $I \subset [r]$, the intersection $X^I := \bigcap_{i \notin I} X^i$ is non-empty, smooth and transverse;

3. every irreducible G-stable subvariety of X has the form X^I for some $I \subset [r]$.

If a wonderful embedding of X_0 exists, then it is unique up to G-equivariant isomorphism and it is G-spherical.

The variety of *complete quadrics*, $\mathcal{X} := \mathcal{X}_n$, is classically defined as the closure of the image of the map

$$\mathcal{X}_0 \to \prod_{i=1}^{n-1} \mathbb{P}(\Lambda^i(\mathrm{Sym}_n)),$$
$$[A] \mapsto ([A], [\Lambda^2(A)], \ldots, [\Lambda^{n-1}(A)]),$$

where A denotes the full rank $n \times n$ symmetric matrix defining a non-singular quadric hypersurface in $\mathcal{X}_0$.

The connection between this classical definition and wonderful embeddings is made in the second half of the following theorem.

Theorem 2.1. 1. [22] $\mathcal{X}_n$ can be obtained by the following sequence of blow-ups: in the naive projective space compactification of $\mathcal{X}_0$, first blow up the locus of rank 1 quadrics; then blow up the strict transform of the rank 2 quadrics; ...; then blow up the strict transform of the rank $n-1$ quadrics.

2. [12] $\mathcal{X}_n$ is the wonderful embedding of $\mathcal{X}_0 := SL_n/SO_n$ and the spherical roots of $\mathcal{X}_n$ are twice the simple positive roots of the type $\mathrm{A_{n-1}}$ root system.

2.3 Orbits in the variety of complete quadrics

A composition of n is an ordered sequence $\mu = (\mu_1, \ldots, \mu_k)$ of positive integers that sum to n. The elements of the sequence are called the *parts* of μ. The compositions of n are in bijection with the subsets of $[n-1]$ via

$$\mu = (\mu_1, \ldots, \mu_k) \longleftrightarrow I(\mu) := [n] \setminus \{\mu_1, \mu_1 + \mu_2, \ldots, \mu_1 + \cdots + \mu_{k-1}\}. \tag{2}$$

The correspondence gives a simple means to define the refinement order on compositions of n. A composition μ of n refines a composition ν of n, denoted $\mu \preceq \nu$, if and only if $I(\mu) \subseteq I(\nu)$. Informally, μ refines ν if μ can be obtained from ν by subdividing its parts. Also note that because of the use of the complement to define $I(\mu)$, the most refined composition, $(1, 1, \ldots, 1)$ is the minimal element under $\preceq$. Using this non-standard bijection to define $I(\mu)$ is convenient for describing the geometry of the variety of complete quadrics.

The SL_n-orbits of $\mathcal{X}_n$ are parameterized by compositions of n; we let $\mathscr{O}^\mu$ denote the associated G-orbit and $\mathcal{X}^\mu$ the closure of $\mathscr{O}^\mu$ in $\mathcal{X}_n$. The refinement order on compositions corresponds to the inclusion order on SL_n-orbit closures: $\mathcal{X}^\mu \subseteq \mathcal{X}^\nu \iff \mu \preceq \nu$. The composition $\mu = (n)$ corresponds to the open SL_n-orbit of smooth quadrics in $\mathbb{P}^{n-1}$, while the composition $\mu = (1, 1, \ldots, 1)$ corresponds to the unique closed SL_n-orbit, isomorphic to the variety of complete flags in the fixed n-dimensional vector space $\mathbb{C}^n$.

Concretely, a complete quadric Q corresponds to giving a partial flag

$$\mathcal{F} : 0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{k-1} \subset V_k = \mathbb{C}^n$$

and for each $1 \leq j \leq k$ a smooth quadric hypersurface in $\mathbb{P}(V_j/V_{j-1})$. Letting $i_j := \dim V_j$, the set $I = [n] \setminus \{i_1, i_2, \ldots, i_{k-1}\}$ corresponds to a composition μ via (2) and $Q \in \mathscr{O}^\mu$. The natural map which sends Q to $\mathcal{F}$ is a morphism $\tilde{p}_\mu : \mathscr{O}^\mu \to \text{Flags}(\mu)$, where $\text{Flags}(\mu)$ is the variety of partial flags whose dimensions are given by μ as above, which extends to a morphism $p_\mu : \mathcal{X}^\mu \to \text{Flags}(\mu)$. The extension is defined by sending a complete quadric Q in $\mathscr{O}^\nu$ first to the flag $\tilde{p}_\nu(Q)$ and then applying the forgetful map $\text{Flags}(\nu) \to \text{Flags}(\mu)$, since ν must be a refinement of μ.

Fix a composition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ of n. We now describe the B-orbits within the SL_n-orbit $\mathscr{O}^\mu$. They are parameterized by combinatorial objects called *μ-involutions*.

Definition 2.2. A *μ-involution* π is a permutation of the set $[n]$, which when written in one-line notation and partitioned into strings of size given by μ, that is $\pi = [\pi_1|\pi_2|\ldots|\pi_k]$ with π_j a string of length μ_j, has the property that each π_j is an involution when viewed as the one-line notation of a permutation of its alphabet. We denote by $\mathcal{I}_\mu$ the set of μ-involutions.

For example, $\pi = [26|8351|7|94]$ is a $(2, 4, 1, 2)$-involution and the string 8351 is viewed as one-line notation for the involution $(1, 8)(3)(5)$ of its alphabet. (We adopt the non-standard convention of including one-cycles when writing a permutation in cycle notation, since we have to keep track of what alphabet is being permuted when working with μ-involutions.)

Definition 2.3. Let $\pi = [\pi_1|\pi_2|\ldots|\pi_k]$ be a μ-involution. Associated to π is a *distinguished complete quadric* Q_π. The flag type of Q_π is given by $I(\mu)$, namely the associated flag

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{k-1} \subset V_k = \mathbb{C}^n$$

has $\dim V_j = \mu_1 + \mu_2 + \cdots + \mu_j$ for $1 \leq j \leq k$. Let $\mathcal{A}_j$ denote the set of numbers that appear in the string π_j. Let $e_1, e_2, \ldots, e_n$ denote the standard basis of $\mathbb{C}^n$. Then V_j is spanned by the vectors e_r for $r \in \bigcup_{i=1}^{j} \mathcal{A}_j$. Note that V_j/V_{j-1} is spanned by the standard basis vectors e_r with $r \in \mathcal{A}_j$. The involution π_j then determines the non-degenerate quadric in $\mathbb{P}(V_j/V_{j-1})$ as in the usual correspondence between involutions and quadrics. Namely, suppose π_j consists of two-cycles $(a_1, b_1), \ldots, (a_s, b_s)$ and one-cycles $c_1, \ldots, c_t$. Then the associated quadric is $x_{a_1}x_{b_1} + \cdots + x_{a_s}x_{b_s} + x_{c_1}^2 + \ldots x_{c_t}^2$.

Returning to the example above and letting $V_{i_1 i_2 \cdots i_s}$ stand for the subspace spanned by $e_{i_1}, e_{i_2}, \ldots, e_{i_s}$, the distinguished quadric Q_π associated to $\pi = [26|8351|7|94]$ consists of the flag

$$0 \subset V_{26} \subset V_{123568} \subset V_{1235678} \subset \mathbb{C}^9$$

and the associated sequence of non-degenerate quadrics is $x_2^2 + x_6^2, x_1x_8 + x_3^2 + x_5^2, x_7^2, x_4x_9$.

The B-orbit through the distinguished quadric Q_π is denoted $\mathcal{O}^\pi$ and its closure is denoted $\mathcal{X}^\pi$. Note that the SL_n-orbit $\mathscr{O}^\mu$ is a disjoint union of B-orbits $\mathcal{O}^\pi$ as π ranges over all μ-involutions.

2.4 Torus actions on spherical varieties

We recall some standard results about reductive algebraic groups [3]. Recall that we set G to denote a reductive group containing a Borel subgroup B and maximal torus $T \subseteq B$. Given any set $A \subset G$, $C_G(A)$ denotes the centralizer of A in G, namely the elements $g \in G$ such that $gag^{-1} = a$ for all $a \in G$.

An element $g \in G$ is called *regular* if $\dim C_G(g)$ is minimal, in which case it equals $\operatorname{rank}(G) := \dim T$. A torus S in G is called regular if S contains a regular element of G and S is called *singular* if S is contained in infinitely many Borel subgroups of G. Since G is reductive, every torus of G is either regular or singular. A maximal torus T of G is always regular, while a maximal singular subtorus S of T necessarily has codimension one and is equal to the kernel of a root $\alpha : T \to \mathbb{C}^*$.

Let $S \subseteq T$ denote a codimension one subgroup, and let $C_G(S)$ denote the centralizer of S, which is a connected reductive group. If S is regular, $C_G(S) = T$, and if S is singular, then $C_G(S)$ has semisimple rank one. If S normalizes B, then $C_B(S)$ is a Borel subgroup of $C_G(S)$ and conversely any Borel subgroup of $C_G(S)$ has the form $C_B(S)$ for a Borel subgroup B that normalizes S.

Suppose that G acts on an algebraic variety X with finitely many orbits. Then the fixed locus X^T is finite [12, §7.3]. For any subtorus $S \subseteq T$, the fixed locus X^S is stable under the action of $C_G(S)$. We now recall structural properties of X^S when S has codimension one [5, 1].

Let $x \in X^S$. Then the intersection of the B-orbit of x with X^S is equal to the $C_B(S)$ orbit of x. Moreover, any irreducible component of X^S is a spherical $C_G(S)$ variety.

If S is a regular torus of codimension one in T, then any irreducible component of X^S is isomorphic to a point or $\mathbb{P}^1$.

Now suppose that S is a singular torus of codimension one in T. Then $C_G(S)$ is the product of T and a subgroup Γ isomorphic to SL_2 or PSL_2. In either case, Γ is the homomorphic image of SL_2, so any irreducible component Y of X^S is a spherical SL_2-variety. Choose a point $y \in Y$ such that $SL_2 \cdot y$ is open in Y, and let $H \subseteq SL_2$ denote the stabilizer subgroup of y. If Y is not an isolated point, then there are four possibilities for H:

1. If H is a Borel subgroup of SL_2, then Y is isomorphic to $\mathbb{P}^1$.
2. If H is a one-dimensional torus in SL_2, then Y is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.
3. If H is the normalizer of a one-dimensional torus in SL_2, then Y is isomorphic to $\mathbb{P}^2$.
4. If H is the product of a one-dimensional unipotent subgroup and a finite cyclic subgroup of a torus in SL_2, then Y is a Hirzebruch surface.[5]

2.5 Weak Order for μ-involutions

Fix a composition $\mu = (\mu_1, \ldots, \mu_k)$ of n. The length of a μ-involution $\pi = [\pi_1|\pi_2|\ldots|\pi_k]$ is defined to be

$$L_\mu(\pi) := \ell(w(\pi)) + \sum_{i=1}^{k} L(\pi_i), \tag{3}$$

where $w(\pi)$ is the permutation obtained by rearranging the elements in each string π_i in increasing order and $L(\pi_i)$ is the length of the corresponding involution π_i as defined by (1). For example, if $\pi = [5326|41]$, then $w(\pi) = 235614$ and $L_{(4,2)}(\pi) = 6 + 2 + 1 = 9$. Let min (resp, max) denote the string $n \ldots 21$ (resp., $12 \ldots n$) partitioned according to μ. These will be the minimum and maximum elements of weak order on μ-involutions defined below.

The Richardson-Springer monoid of S_n, $M(S_n)$, is defined to be the monoid generated by elements $s_1, \ldots, s_{n-1}$ subject to the relations that $s_i^2 = s_i$ for all i, $s_i s_j = s_j s_i$ if $|i - j| > 1$, and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $1 \leq i < n - 1$. (Note the notational ambiguity of s_i standing for the simple transposition in S_n interchanging i and $i + 1$ and the generator of $M(S_n)$. It should be clear from context which meaning is meant in the sequel.) The elements of $M(S_n)$ are in bijection with the elements of S_n, with the element $w \in S_n$ associated to the element of $M(S_n)$ given by $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ for any reduced decomposition of S_n.

There is a natural action of the Richardson-Springer monoid of S_n on the set of all B-orbits in $\mathscr{O}^\mu$, and consequently on the set of all μ-involutions [18]. In the case where π is an ordinary involution of S_n (i.e. when $\mu = (n)$), the action of a simple transposition s_i interchanging i and $i + 1$ is given by

$$s_i \cdot \pi = \begin{cases} s_i \pi s_i & \text{if } \ell(s_i \pi s_i) = \ell(\pi) - 2 \\ s_i \pi & \text{if } s_i \pi s_i = \pi \text{ and } \ell(s_i \pi) = \ell(\pi) - 1 \; . \\ \pi & \text{otherwise} \end{cases} \tag{4}$$

If $\pi = [\pi_1|\pi_2|\ldots|\pi_k]$ is a general μ-involution, then there are two cases for the computation of $s_i \cdot \pi$. If i and $i + 1$ belong to the same string π_r of π, then $s_i \cdot \pi$ is obtained by fixing all other π_j's and replacing π_r with $s_i \cdot \pi_r$ as defined by (4). If i and $i + 1$ belong to different strings then $s_i \cdot \pi$ is equal to the result of interchanging i and $i + 1$ if $i + 1$ precedes i in π and is equal to π otherwise.

To define $w \cdot \pi$ for an arbitrary $w \in S_n$ and arbitrary μ-involution π, let $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ be any reduced decomposition of w. Then $w \cdot \pi := s_{i_1} \cdot (s_{i_2} \cdots (s_{i_\ell} \cdot \pi) \cdots)$. Because the action of simple transpositions respects the braid relations associated to the Richardson-Springer monoid of S_n, it is easily checked that this definition is independent of the choice of reduced decomposition.

Let G be a complex reductive algebraic group with Borel subgroup B, and let X be a spherical G-variety. Let Y be an irreducible B-stable subvariety of X and let $w \in W$ be a Weyl group element. The map

$$\overline{BwB} \times Y \to \overline{BwY}$$
$$(g, y) \mapsto gy$$

is invariant with respect to the action of B defined by $b \cdot (g, y) = (gb^{-1}, by)$. Let $\overline{BwB} \times_B Y$ denote the corresponding quotient and let

$$\pi_{Y,w} : \overline{BwB} \times_B Y \to \overline{BwY}$$

denote the induced morphism $[(g, y)] \mapsto gy$. Note that since $\overline{BwB}/B$ is complete, $\pi_{Y,w}$ is proper, hence surjective. The W-set of Y is defined to be

$$W(Y) := \{w \in W : \ \pi_{Y,w} \ \text{ is generically finite, } \ \overline{BwY} \ \text{ is } G\text{-invariant}\}. \tag{5}$$

There is a more combinatorial interpretation of W-sets which essentially follows from [6, Lemma 1.2]. For two μ-involutions π and ρ, $\pi \leq \rho$ in weak order if and only if $\rho = w \cdot \pi$ for some w in the Richardson-Springer monoid of S_n. Covering relations in weak order are labeled by simple roots. By definition, a B-orbit closure Y' covers another B-orbit closure Y in the weak order if there exists a minimal parabolic subgroup $P = P_\alpha$ corresponding to a simple root α such that $Y' = PY$. The label of this covering is α. For each maximal chain starting at a μ-involution π ending at max in weak order there is a Weyl group element that is obtained by multiplying in order, right to left, the simple reflections corresponding to the labels (starting at π). The set of all $w \in W$ obtained this way constitutes the W-set of π, denoted $W(\pi)$ [8, 9, 10]. In fact, we can gave an explicit description of a slightly more general definition.

Definition 2.4. Let π, ρ be a μ-involution and let max (resp. min) denote the maximal (resp. minimal) elements in the weak order poset of μ-involutions. Then

$$W(\pi, \rho) := \{w \in S_n : w \cdot \pi = \rho \text{ and } \ell(w) = L_\mu(\pi) - L_\mu(\rho)\}.$$

In particular, we call $W(\pi) := W(\pi, \max)$ the *W-set* of π and $W^{-1}(\pi) := W(\min, \pi)$ the *reverse W-set* of π.

The inverse of an element in $W(\pi, \rho)$ in Definition 2.4 is referred to as an *atom* and the set of atoms are described concretely by Theorems 5.10 and 5.11 in [14].

2.6 Białynicki-Birula Decomposition

We recall the Białynicki-Birula decomposition, or BB decomposition for short, associated to a smooth projective variety with the action of a one-parameter subgroup. Let X be a smooth projective variety on which an algebraic torus T acts with finitely many fixed points. Let $\lambda : \mathbb{C}^* \to T$ be a one-parameter subgroup such that $X^\lambda = X^T$. For $p \in X^\lambda$, define the sets

$$C_p^+ = \{y \in X : \lim_{s \to 0} \lambda(s) \cdot y = p\}$$

and

$$C_p^- = \{y \in X : \lim_{s \to \infty} \lambda(s) \cdot y = p\},$$

called the plus and minus cells of p, respectively.

Theorem 2.5 ([2])**.** If X, T and λ are as above, then

1. both of the sets C_p^+ and C_p^- are locally closed subvarieties in X, furthermore they are isomorphic to an affine space;
2. if T_pX is the tangent space of X at p, then C_p^+ (resp., C_p^-) is λ-equivariantly isomorphic to the subspace T_p^+X (resp., T_p^-X) of T_pX spanned by the positive (resp., negative) weight spaces of the action of λ on T_pX.

As a consequence of the BB-decomposition, there exists a filtration

$$X^\lambda = V_0 \subset V_1 \subset \cdots \subset V_n = X, \qquad n = \dim X,$$

of closed subsets such that for each $i = 1, \dots, n$, $V_i \setminus V_{i-1}$ is the disjoint union of the plus (resp., minus) cells in X of dimension i. It follows that the odd-dimensional integral cohomology groups of X vanish, the even-dimensional integral cohomology groups of X are free, and the Poincaré polynomial

$$P_X(t) := \sum_{i=0}^{2n} \dim H^i(X; \mathbb{C}) t^i$$

of X is given by

$$P_X(t) = \sum_{p \in X^\lambda} t^{2 \dim C_p^+} = \sum_{p \in X^\lambda} t^{2 \dim C_p^-}.$$

3 Barred permutations

In this section we give a parametrization of the torus fixed points in X.

Definition 3.1. A composition μ of n is *special* if every part of μ is either 1 or 2. Equivalently, μ is special if the associated subset $I(\mu)$ does not contain any consecutive numbers.

Note that any refinement of a special composition is also a special composition. Special compositions are used to characterize the SL_n-orbits of $\mathcal{X}_n$ which contain T-fixed points. Indeed, a composition μ is special if and only if the orbit $\mathscr{O}^\mu$ contains a T-fixed point [20].

Since an orbit of B with isolated T-fixed points has exactly one T-fixed point, in order to parameterize the T-fixed points in $\mathcal{X}_n$, it suffices to make explicit the μ-involutions of those B-orbits which contain a T-fixed point.

Definition 3.2. Let μ be a composition of n with k parts. We call a μ-involution $\alpha = [\alpha_1 | \cdots | \alpha_k]$ a *barred permutation* if the length of each α_j (as a string) is at most 2 and whenever $\alpha_j = i_1 i_2$ has length two, then $i_1 > i_2$. The length of α is n. We let $\mathcal{B}_n$ denote the set of barred permutations of length n and put $b_n := \#\mathcal{B}_n$.

Lemma 3.3. Let μ be a composition of n and let π be a μ-involution. Then the B-orbit $\mathcal{O}^\pi$ contains a T-fixed point if and only if π is a barred permutation, in which case the T-fixed point is the distinguished quadric Q_π.

Proof. Let $\mu = (\mu_1, \ldots, \mu_k)$ be the given composition of n, and let e_i denote the ith standard basis vector of $\mathbb{C}^n$. We already know that when there is a T-fixed point in the B-orbit $\mathcal{O}^\pi$ for some μ-involution $\pi = [\pi_1|\cdots|\pi_k]$, the length of each string in π is at most 2.

Let Q denote a T-fixed complete quadric belonging $\mathcal{O}^\pi$, consisting of a flag

$$\mathcal{F} : 0 = V_0 \subset V_1 \subset \cdots \subset V_{k-1} \subset V_k = \mathbb{C}^n$$

of type μ (meaning $\dim V_j/V_{j-1} = \mu_j$) and a sequence $Q_1, \ldots, Q_k$ of smooth quadrics in $\mathbb{P}(V_j/V_{j-1})$. Since the morphism $\mathscr{O}^\mu \to \mathrm{Flag}(\mu)$ is SL_n-equivariant, if Q_π is T-fixed, $\mathcal{F}$ must also be T-fixed. Thus, each V_j/V_{j-1} is spanned by (the projection of) the standard basis vectors e_i for the elements $i \in [n]$ that occur in the string π_j. The action of T on the quadric Q_j reduces to the action of the diagonal matrices in SL_{μ_j} on full rank $\mu_j \times \mu_j$ symmetric matrices.

It is a straightforward exercise to check that the only smooth quadric hypersurfaces in $\mathbb{P}^n$ that are T-fixed are the quadric hypersurfaces $x_1^2 = 0$ when $n = 1$ and $x_1 x_2 = 0$ when $n = 2$. Therefore, if $\mathcal{O}^\pi$ contains a T-fixed complete quadric, then each $\mu_j \leq 2$ and if $\mu_j = 2$, then Q_j must correspond to the symmetric matrix which interchanges the two variables, so if $\pi_j = i_1 i_2$, then we must have $i_1 > i_2$ in order for Q to belong to $\mathcal{O}^\pi$. Thus, π must be a barred permutation and Q must be the distinguished quadric Q_π. Moreover, if π is a barred permutation, then Q_π is indeed T-fixed. □

In the rest of the section we determine the number of T-fixed points in $\mathcal{X}$.

Lemma 3.4. The sequence b_n satisfies the recurrence given by

$$b_{n+1} = \binom{n+1}{2} b_{n-1} + (n+1) b_n \text{ for } n \geq 1, \tag{6}$$

and the initial conditions $b_0 = b_1 = 1$.

Proof. Let $\pi = [\pi_1|\cdots|\pi_{k-1}|\pi_k]$ be a barred permutation on $[n+1]$ and count possibilities for π according to its last string π_k. The first term in the recurrence counts the number of barred permutations where the length of π_k is 2 and the second term in the recurrence counts the number of barred permutations where the length of π_k is 1. □

The simple substitution $a_k := b_k/k!$ converts the recurrence (6) into a linear recurrence,

$$a_k = a_{k-1} + \frac{1}{2} a_{k-2} \text{ for } k \geq 2 \tag{7}$$

with initial conditions $a_0 = a_1 = 1$.

By standard techniques, the generating series for the sequence a_k is

$$A(x) := \sum_{k=0}^{\infty} a_k x^k = \frac{1}{1 - x - x^2/2}.$$

Proposition 3.5. The exponential generating series $F(x) = \sum_{n\geq 0} \frac{b_n}{n!} x^n$ for the number of barred permutations of length n is given by

$$F(x) = \frac{1}{1 - x - x^2/2} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{3}} \left(\frac{(1+\sqrt{3})^{n+1} - (1-\sqrt{3})^{n+1}}{2^{n+1}} \right) x^n.$$

Therefore, the number of T-fixed points in $\mathcal{X}_n$ is

$$b_n = n! a_n = \frac{n!}{2^n} \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2i+1} 3^i.$$

4 T-stable curves and surfaces in the variety of complete quadrics

Let $S \subset T \subset SL_n$ be a singular codimension one subtorus of T, the kernel of a simple positive root α, and let Y denote an irreducible component of $\mathcal{X}_n^S$, the subvariety of S-fixed complete quadrics. Let U_α and $U_{-\alpha}$ denote the associated one-dimensional unipotent subgroups of SL_n associated to α and $-\alpha$, and let $\Gamma_\alpha \cong SL_2$ be the subgroup of SL_n generated by U_α and $U_{-\alpha}$.

Let P_μ denote the standard parabolic subgroup associated with $\mathscr{O}^\mu$, that is the stabilizer of the standard flag in the variety Flags(μ). Then P_μ has a Levi decomposition as $L_\mu \ltimes U_\mu$, where U_μ is a maximal unipotent subgroup of P_μ and L_μ is the Levi subgroup of P_μ. Note that a Levi subgroup of SL_n has the form $S\left(\prod_{i=1}^{k} GL_{\mu_i}\right)$, for some composition μ of n. Here the factors GL_{μ_i} are embedded block diagonally in SL_n and the $S(\cdot)$ notation refers to the subgroup of elements of determinant 1.

Proposition 4.1. Let μ be a composition and let id_μ denote the identity μ-involution, namely, the string $12\ldots n$ partitioned according to μ. The stabilizer subgroup of Q_{id_μ} in SL_n is equal to

$$S(\widetilde{SO}_{\mu_1} \times \cdots \times \widetilde{SO}_{\mu_k}) \ltimes U_\mu, \tag{8}$$

where $\widetilde{SO}_{\mu_i}$ is the normalizer of SO_{μ_i} in GL_{μ_i}, or equivalently the central extension of SO_{μ_i} in GL_{μ_i}.

Proof. The flag associated with Q_{id_μ} is the standard flag in Flags(μ), whose stabilizer is P_μ. Because the individual quadrics in the complete quadric Q_{id_μ} lie on the successive quotient spaces in the standard flag, the action on each the ith quadric is determined by the ith diagonal block, and since each individual quadric corresponds to the identity symmetric matrix (standard bilinear form), its stabilizer is the central extension of the orthogonal subgroup of the ith diagonal block. $\square$

Proposition 4.2. Let $\alpha = \alpha_{i,j}$, $i < j$, be the positive root of SL_n such that $\alpha(\mathrm{diag}(t_1, \ldots, t_n)) = t_i t_j^{-1}$. Let $\beta = [\beta_1 | \cdots | \beta_k]$ be a barred permutation, and Q_β the associated distinguished (and T-fixed) complete quadric. Then the stabilizer of the action of Γ_α at Q_β is isomorphic to

1. SO_2, or equivalently the normalizer of the maximal torus of Γ_α, if i and j belong to the same length 2 string in β, or

2. a Borel subgroup of Γ_α, if i and j belong to different strings in β

Proof. If i and j belong to the same string of β, then every element of Γ_α stabilizes the underlying flag of Q_β and the action on the quadric $x_i x_j$ is the usual SL_2 action on non-degenerate quadrics in two variables. It follows that the stabilizer in this case is isomorphic to SO_2. If i and j belong to different strings, then a Borel subgroup of Γ_α fixes the underlying flag of Q_β, and then those elements also fix each quadric since the variables x_i and x_j do not occur in the same quadric. □

Theorem 4.3. Let S be a codimension one subtorus of T. Any irreducible component of $\mathcal{X}_n^S$ is isomorphic to a point, $\mathbb{P}^1$ or $\mathbb{P}^2$.

Proof. If S is a regular codimension-one subtorus, then as noted in Section 2.4, any irreducible component of $\mathcal{X}_n^S$ is isomorphic to a point or $\mathbb{P}^1$.

Suppose that S is a singular codimension-one subtorus of T, the kernel of the simple root $\alpha : T \to \mathbb{C}^*$ and let Y denote an irreducible component of $\mathcal{X}_n^S$. By Proposition 4.2, the isotropy subgroup of a generic point from the open SL_2-orbit in Y is isomorphic to SO_2 or a Borel subgroup B' of SL_2. In the former case $Y \cong \mathbb{P}^2$, while in the latter case, $Y \cong \mathbb{P}^1$. □

When SL_2 acts on $\mathbb{P}^2$ via its standard three-dimensional representation, which we identify with $\mathrm{Sym}^2(\mathbb{C}^2)$, 2×2 symmetric matrices, there are two SL_2-orbits; the open orbit corresponding to non-singular quadrics and the closed orbit corresponding to degenerate quadrics. The latter orbit is isomorphic to $\mathbb{P}^1$, the flag variety of SL_2. Let $T' \subset SL_2$ denote the maximal torus consisting of diagonal matrices and let B' denote the Borel subgroup of upper triangular matrices. The dense SL_2-orbit has two B'-orbits: the orbit of the quadric defined by $x^2 + y^2$ which does not have any T'-fixed points, and the orbit of the quadric defined by xy, which is a T'-fixed quadric. The closed SL_2-orbit has two T'-fixed points.

Remark 4.4. It follows that if $Y \subset \mathcal{X}_n$ is isomorphic to $\mathbb{P}^2$ and invariant under the maximal torus T of SL_n, then the SL_2-closed orbit in Y is stable under T.

Lemma 4.5. Let Y denote an irreducible component of $\mathcal{X}^S$, where S is a codimension-one subtorus of T. Assume that Y contains at least two T-fixed points. If Y is contained in a G-orbit $\mathscr{O}^I$, then Y contains exactly two T-fixed points.

Proof. Recall that each G-orbit $\mathscr{O}^\mu$ has a G-equivariant fibration over a flag variety G/P_μ with fibers isomorphic to $S(\prod_{i=1}^k GL_{\mu_k})/S(\prod_{i=1}^k \widetilde{SO}_{\mu_k})$. Assuming that $\mathscr{O}^\mu$ contains a T-fixed point, we have each $\mu_k = 1$ or 2. Identifying SL_2/SO_2 with the space of all nonsingular quadrics in $\mathbb{C}^2$, we observe that there is exactly one $\mathbb{C}^*$-fixed point in SL_2/SO_2, hence there is at most one T-fixed point on each fiber of the fibration $\pi_\mu : \mathscr{O}^\mu \to G/P_\mu$. Therefore, under π_μ the set of T-fixed points in Y is mapped bijectively onto the set of T-fixed point contained in $Y' := \pi_\mu(Y) \subset G/P_\mu$. Since Y' is S-fixed in G/P_μ, Y' either is a single point or is isomorphic to $\mathbb{P}^1$. Since we assumed Y has at least two T-fixed points, Y' and hence Y has exactly two T-fixed points. □

Corollary 4.6. Let Y denote an irreducible component of $\mathcal{X}^S$, where S is a codimension-one subtorus of T. Assume that Y contains exactly three T-fixed points $x, y, z \in Y \cap \mathcal{X}^T$. In this case, Y intersects two G-orbits $\mathscr{O}^\mu$ and $\mathscr{O}^\nu$ such that (without loss of generality)

1. $y \in \mathscr{O}^\mu$ and $x, z \in \mathscr{O}^\nu$;
2. $I(\mu) \subset I(\nu)$, hence $\mathscr{O}^\nu \subset \mathcal{X}^\mu$. Furthermore, $\dim \mathscr{O}^\mu = \dim \mathscr{O}^\nu + 1$.

Proof. Since Y contains three T-fixed points, we know that Y is T-equivariantly isomorphic to $\mathbb{P}^2$. We know also from Lemma 4.5 that Y intersects (at least) two G-orbits, which we denote by $\mathscr{O}^\mu$ and $\mathscr{O}^\nu$. Without loss of generality we assume that $\mathscr{O}^\mu$ is the SL_n-orbit that contains $y \in Y$ for which $SL_2 \cdot y$ is open in Y. By Remark 4.4, the closed orbit in Y is T-equivariantly isomorphic to $\mathbb{P}^1$, hence it contains exactly two T-fixed points. Since these two fixed points are in the same SL_2-orbit, they are in the same SL_n-orbit.

We know from the proof of Proposition 4.2 that there is a subgroup $\Gamma \subseteq SL_n$ isomorphic to SL_2 that acts on the complete quadric y in such a way that the flag is always fixed and the action on the sequence of quadrics acts on a single quadric. If it is the ith quadric, then $\mu_i = 2$. Let the barred permutation associated to y be $\beta = [\beta_1|\dots|\beta_i|\dots|\beta_k]$, with $\beta_i = ba$ being a string of length two with $a < b$. Then limiting T-fixed points x and z lie in $\mathcal{O}^\nu$ where

$$\nu = (\mu_1, \dots, \mu_{i-1}, 1, 1, \mu_{i+1}, \dots, \mu_k).$$

In fact the barred permutations associated to x and z are $\beta' = [\beta_1|\dots|a|b|\dots|\beta_k]$ and $\beta'' = [\beta_1|\dots|b|a|\dots|\beta_k]$. The second assertion follows from this observation in the light of the isomorphism between the refinement partial ordering on compositions and the inclusion ordering on SL_n-orbit closures in $\mathcal{X}_n$. □

Example 4.7. Consider the $\mu = (2,1)$-involution $\alpha = [21|3]$ having a T-fixed point. The SL_2-orbit closure in this case has three T-fixed points whose μ-involutions are $\beta_1 = \alpha$, $\beta_2 = [2|1|3]$, and $\beta_3 = [1|2|3]$.

Suppose we have two T-fixed points denoted by x and y contained in the B-orbits $\mathcal{O}^\alpha \subseteq \mathscr{O}^\mu$ and $\mathcal{O}^\beta \subseteq \mathscr{O}^\nu$, respectively. Assume for definiteness that $\mathcal{X}^\beta \subseteq \mathcal{X}^\alpha$. There are two cases:

1. $\mu = \nu$. In this case, α is obtained from β by applying a transposition: there exists a pair of numbers $1 \le i < j \le n$ such that α is obtained from β by interchanging i and j. For example, $\alpha = [31|2]$ and $\beta = [21|3]$.
2. $\mu \neq \nu$. In this case, one of the 2-strings of α is split into two 1-strings in β. For example, $\alpha = [21|3]$ and $\beta = [2|1|3]$, or $\alpha = [21|3]$ and $\beta = [1|2|3]$.

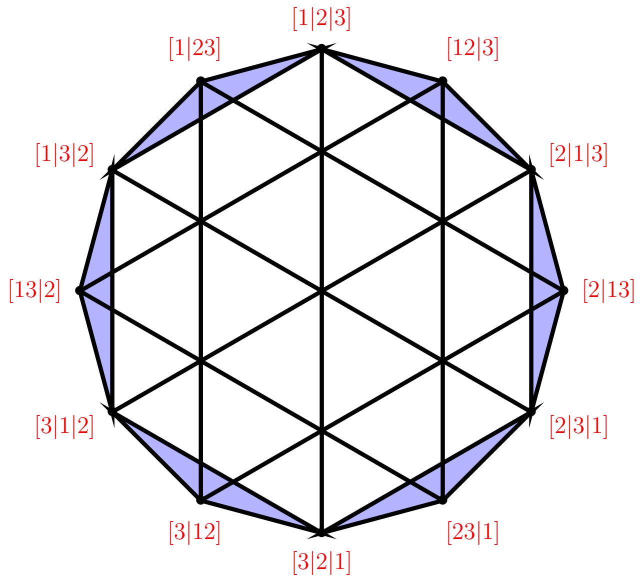

Figure 1: T-stable curves and surfaces in $\mathcal{X}_3$.

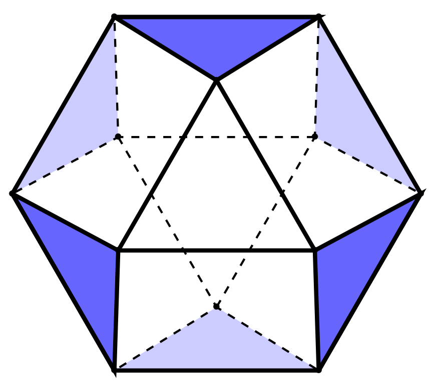

Figure 2: The moment polytope associated to $\mathcal{X}_3$, a cubeoctahedron. Shaded triangles correspond to the T-invariant $\mathbb{P}^2$'s.

Example 4.8. Figure 1 is the graph of all T-invariant surfaces and T-invariant curves in $\mathcal{X}_3$; the shaded triangles represent T-stable $\mathbb{P}^2$'s, while the edges indicate T-stable $\mathbb{P}^1$'s. The graph is shown in Figure 2 embedded in the moment polytope associated to $\mathcal{X}_3$.

Now that we have a full characterization of T-fixed points, T-stable curves and T-stable surfaces in $\mathcal{X}_n$, we are ready to provide an explicit description of $K_{T,*}(\mathcal{X}_n)$ following [1]. Let $X(T)$ denote the character group of T and $R(T) = \mathbb{Z}[X(T)]$ denote the representation ring of T, which is generated as a $\mathbb{Z}$-algebra by the elements corresponding to a set of simple roots.

Theorem 4.9. The T equivariant K-theory $K_{T,*}(\mathcal{X}_n)$ is isomorphic to the ring consisting of tuples $(f_x) \in \prod_{x \in \mathcal{B}_n} R(T)$ satisfying the following two congruence conditions:

1. $f_x - f_y = 0 \mod (1-\chi)$ when x, y are connected by a T stable curve with weight χ.
2. $f_x - f_y = f_y - f_z = 0 \mod (1-\chi)$ and $f_x - f_z = 0 \mod (1-\chi^2)$, when x, y, and z lie on a T-stable surface that is fixed by a singular torus which is the kernel of a root χ, with x, y, and z as in Corollary 4.6.

Moreover, the symmetric group S_n acts on the T-fixed points $\mathcal{B}_n$ by the natural action on barred permutations and the G-equivariant K-theory $K_{G,*}(\mathcal{X}_n)$ is equal to the subring of S_n-invariants in $K_{T,*}(\mathcal{X})$.

5 Bruhat order for degenerate involutions

5.1 Bruhat order for $\mathcal{O}^\mu$

Fix a composition $\mu = (\mu_1, \dots, \mu_k)$ of n. Recall that for any μ-involution π, the B-orbit containing Q_π (resp. its closure) is denoted by $\mathcal{O}^\pi$ (resp., $\mathcal{X}^\pi$). The *Bruhat order* on μ-involutions is defined by $\pi \le \pi' \iff \mathcal{X}^\pi \subseteq \mathcal{X}^{\pi'}$. Recall that the length of a μ-involution is defined by (3). Bruhat order on μ-involutions is a ranked poset, with

$$\mathrm{rank}(\pi) := L_\mu(\min) - L_\mu(\pi).$$

We now apply a recursive characterization of the Bruhat order on spherical varieties due to Richardson-Springer [18] and Timashev [21] in the special case of μ-involutions.

Proposition 5.1. Let π and ρ be μ-involutions. Then $\pi \le \rho$ in Bruhat order if and only if

1. $\pi = \rho$; or
2. there exist μ-involutions π^*, ρ^* and a simple transposition s_i such that $\rho = s_i \cdot \rho^*$ with $\rho \neq \rho^*$, $\pi = \pi^*$ or $\pi = s_i \cdot \pi^*$, and $\pi^* \le \rho^*$.

Another equivalent description of Bruhat order is as follows. Let π, ρ be μ-involutions. Then $\pi \lessdot \rho$ is a covering relation in Bruhat order if and only if there exist $v \in S_n$, a simple transposition s_j, and μ-involutions π^*, ρ^* such that $\pi = v \cdot \pi^*$, $\rho = v \cdot \rho^*$, $L(\pi) = L(\pi^*) + \ell(v)$, $L(\rho) = L(\rho^*) + \ell(v)$ and $\rho^* = s_j \cdot \pi^* \neq \pi^*$.

The draw-back of this description of the Bruhat order is that it is built recursively starting with the smallest element of the poset. It is desirable to have a faster, more intrinsic method for checking if two μ-involutions are comparable or not.

Remark 5.2. When $\mu = (1, 1, \ldots, 1)$ (resp., $\mu = (n)$), the Bruhat order on μ-involutions is equal to the opposite of the Bruhat order on S_n (resp., the opposite of the restriction of Bruhat order to the set of involutions in S_n). Unfortunately, there is no such general property for arbitrary compositions. Each μ-involution can be naturally identified with a permutation in S_n by ignoring the division into strings given by μ and considering the string as the one-line notation of a permutation of $[n] = \{1, 2, \ldots, n\}$. The Bruhat order on μ-involutions is graded because Bruhat order for B-orbits in projective spherical variety is always graded [17], but Figure 3 shows that the induced Bruhat order on $(3, 1)$-involutions is not graded. (Consider the interval from $[432|1]$ to $[321|4]$ in the bottom right portion of the figure.)

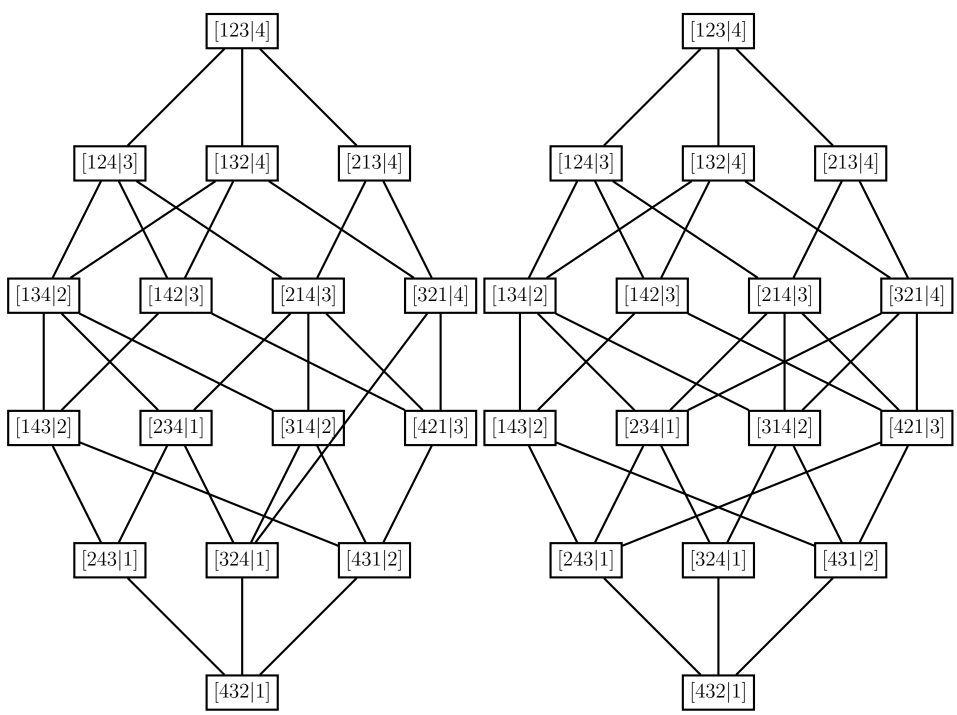

Figure 3: The combinatorial Bruhat order on $(3, 1)$-involutions, on the left, is not graded. The Bruhat order on $(3, 1)$-involutions is on the right.

5.2 Bruhat order for $\mathcal{X}_n$

We now discuss comparing two arbitrary degenerate involutions in Bruhat order, or equivalently determine if $\mathcal{X}^\pi \subseteq \mathcal{X}^\rho$ for arbitrary degenerate involutions π and ρ. To do so, we make use of the technology of W-sets from [6]. We first begin with some general remarks.

Let G be a complex reductive algebraic group with Borel subgroup B containing a maximal torus T. Let Δ be the set of simple roots of the associated root system. If $\alpha \in \Delta$, the minimal parabolic subgroup $B \cup Bs_\alpha B$ is denoted by P_α. If X is a G-variety, the set of all irreducible B-stable subvarieties in X is denoted by $\mathcal{B}(X)$. Recall that X is said to be spherical if B has a dense orbit in X, or equivalently if there are only finitely many B-orbits in X. In that case, $\mathcal{B}(X)$ consists of the closures of B-orbits in X.

Definition 5.3. A G-action on a spherical variety X is called *cancellative* if for any $Y_1, Y_2 \in \mathcal{B}(X)$, distinct from each other, and for any simple root $\alpha \in \Delta$ such that $P_\alpha Y_1 \neq Y_1$ and $P_\alpha Y_2 \neq Y_2$ we have $P_\alpha Y_1 \neq P_\alpha Y_2$.

Remark 5.4. The G-action on generalized flag varieties is cancellative, as is the $G\times G$-action on G. However, the diagonal action of $G = SL_2$ on $\mathbb{P}^1 \times \mathbb{P}^1$ is not [6].

Let P be a parabolic subgroup with Levi subgroup $L \supset T$, X' be an L-variety and let X denote the quotient variety obtained from $G \times X'$ by the action of P, where P acts on G on by right multiplication and it acts on the left on X' via its quotient group L. The variety X is a G-variety by left multiplication on the first factor. We denote X by $G \times_P X'$ and identify X' with the P-invariant subvariety $P \times_P X'$, which is exactly the fiber at P/P of the natural projection $p : G \times_P X' \to G/P$. By Lemma 1.2 of [6], the G-action on the induced variety X is cancellative if and only if the L-action on X' is. Further, if G_1, G_2 are reductive groups and X_1 (resp., X_2) is a cancellative spherical G_1 (resp., G_2) variety, then $X_1 \times X_2$ is a cancellative spherical $G_1 \times G_2$-variety.

Proposition 5.5. For any composition μ, the action of SL_n on $\mathcal{O}^\mu$ is cancellative.

Proof. When $\mu = (n)$, the claim amounts to checking that the weak order on involutions is cancellative, which is immediate from the definition of weak order on involutions. For any other composition $\mu = (\mu_1, \dots, \mu_k)$ of n, the variety $\mathcal{X}^\mu$ is induced from an L-variety X' where $L = S(\prod_{i=1}^k GL_{\mu_i})$ and $X' = \prod_{i=1}^k \mathcal{X}_{\mu_i}$. Moreover, $\mathcal{O}^\mu$ is induced from the dense orbit of X', which is $\prod_{i=1}^k \mathcal{O}^{(\mu_i)}$. Since the SL_{μ_i}-action on $\mathcal{O}^{(\mu_i)}$ is cancellative, it follows by the two remarks prior to this proposition that $\mathcal{O}^\mu$ is cancellative. □

Cancellativeness is a useful property for studying Bruhat orders for several reasons. First of all, since μ-involutions have a unique minimal and a unique maximal elements, the cancellativeness property gives a well-defined action of the opposite Richardson-Springer monoid on μ-involutions. In particular, the resulting weak order is the opposite of the weak order that is defined by the ordinary action of Richardson-Springer monoid. Another application of cancellativeness arises in the context of W-sets. By Theorem 1.4 of [6], if $Y = \mathcal{X}^\pi$ for a degenerate involution π is the closure of a B-orbit in $\mathcal{X}_n$ and $Z = \mathcal{X}^\mu$ for a composition μ of n is the closure of an SL_n-orbit in $\mathcal{X}_n$, then $Y \cap Z$ is the union of $\mathcal{X}^\rho$ for μ-involutions ρ such that $W(\rho)$ is contained in (equivalently, has non-empty intersection with) $W(\pi)$.

Theorem 5.6. Let π be a μ-involution and ρ be a ν-involution and assume $\nu \precneqq \mu$. Then $\rho \leq \pi$ if and only if there exists a ν-involution γ with $W(\gamma) \subseteq W(\pi)$ (equivalently $W(\gamma) \cap W(\pi) \neq \emptyset$) and $\rho \leq \gamma$.

Proof. Let X denote the G-orbit closure associated to μ and Y the G-orbit closure associated to ν. Let X' denote the B-orbit closure associated to π and Y' the B-orbit closure associated to ρ. By Brion's theorem, $X' \cap Y$ is equal to the union of all B-orbit closures Z' associated to ν-involutions γ with $W(\gamma) \subseteq W(\pi)$ (equivalently, $W(\gamma) \cap W(\pi) \neq \emptyset$). (Note that a necessary condition for this inclusion to hold is that $\operatorname{codim}(Z', Y) = \operatorname{codim}(X', X)$.) Thus, if $Y' \subseteq X'$, then since $Y' \subset Y$ and Y' is irreducible, we must have $Y' \subseteq Z'$ for some such Z'. This proves the ($\Rightarrow$) direction.

The ($\Leftarrow$) direction is trivial, for if $Y' \subseteq Z'$ for some $Z' \subseteq X' \cap Y$, then clearly $Y' \subseteq X'$, i.e $\rho \leq \pi$. □

In Timashev's characterization of the Bruhat order (Proposition 5.1), one can start with weak order covering relations in a G-orbit and build covering relations of Bruhat order 'upward' recursively. We define the *reverse Bruhat* order by the same principle but starting from the reverse weak order and moving 'downward' recursively.

Definition 5.7. If π and ρ are two μ-involutions, then π covers ρ in reverse Bruhat order if and only if either ρ covers π in the weak order, or there exist μ-involutions π^*, ρ^* and a simple transposition s_i such that

- $\rho = s_i \star \rho^*$ with $L_\mu(\rho^*) = L_\mu(\rho) + 1$,
- $\pi = s_i \star \pi^*$ and $L_\mu(\pi^*) = L_\mu(\pi) + 1$,

and π^* covers ρ^* in reverse Bruhat order.

Theorem 5.8. The opposite of the reverse Bruhat order on μ-involutions is equal to the Bruhat order on μ-involutions.

Proof. Let $\pi \lessdot \rho$ be a covering in the ordinary Bruhat order. We are going to show that π covers ρ in the reverse Bruhat order.

Without loss of generality we assume that the rank of π is biggest possible. In other words, we assume that if π' and ρ' are two μ-involutions with

- ρ' covers π' in the Bruhat order,
- there exists a simple reflection s such that $s \cdot \pi'$ covers π', $s \cdot \rho'$ covers ρ', and
- $L(\pi') > L(\pi)$,

then the Bruhat cover $s \cdot \pi' \lessdot s \cdot \rho'$ is a reverse Bruhat order cover.

By Timashev's characterization we know that there exists a simple reflection s_{i_1} and μ-involutions π_1^*, ρ_1^* such that $s_{i_1} \cdot \pi_1^* = \pi$ and $\pi_1^* \neq \pi$, $s_{i_1} \cdot \rho_1^* = \rho$ and $\rho_1^* \neq \rho$, and π_1^* is covered by ρ_1^* in Bruhat order. If the last covering relation is not a weak order cover, by Timashev's characterization once again, we know that there exists a simple reflection s_{i_2} ($i_1 \neq i_2$) and μ-involutions π_2^*, ρ_2^* such that $s_{i_2} \cdot \pi_2^* = \pi_1^*$, $s_{i_2} \cdot \rho_2^* = \rho_1^*$, and π_2^* is covered by ρ_2^* in Bruhat order. And this procedure continues until we reach at a weak order covering relation $\pi_m^* \lessdot \rho_m^*$ via $s_j \cdot \pi_m^* = \rho_m^*$ for some simple reflection s_j. For notation, we set $\pi_l^* = s_{i_{l+1}} \cdot \pi_{l+1}^*$, $\rho_l^* = s_{i_{l+1}} \cdot \rho_{l+1}^*$ for $l = 0, 1, \ldots, m-1$.

Let u be an element of the W-set of the interval $[\pi, \max]$ and suppose $s_{j_1} s_{j_2} \cdots s_{j_k}$ is a reduced expression for u. Hence, the corank of π is k. Clearly, the product

$$w := s_{i_m} \cdots s_{i_1} s_{j_1} \cdots s_{j_k} \tag{9}$$

is a member of the reverse W-set of the interval $[\max, \pi_m^*]$. It is also clear that $s_j \star \rho_m^* = \pi^*$. Then applying [18, PROPERTY 5.12(e) (Exchange Property)] to opposite weak order (with $x = \pi_m^*$, $y = \rho_m^*$, $s := s_j$) gives us a reverse W-set element $w' \in W([\max, \rho_m^*])$ that is obtained from w by omitting one of the simple reflections, denoted by s_m, in (9). We depict what we have so far in Figure 4, where $\widetilde{s_{j_l}} = s_{j_l}$ for $j_l \neq m$.

Note that if s_m is one $\{s_{i_1}, \ldots, s_{i_m}\}$, then we get a contradiction with the cancellativeness property since we already know that $\rho = s_{i_1} \cdot s_{i_2} \cdot \cdots \cdot s_{i_m} \cdot \rho_m^*$. On the other hand, if $s_m \in \{s_{j_1}, \ldots, s_{j_k}\}$, then by (Timashev's) construction of the reverse Bruhat order, we obtain a covering reverse Bruhat order covering relation between

$$\begin{cases} s_{j_k} \star \cdots \star s_{j_{m-1}} \star \max \quad \text{and} \quad s_{j_k} \star \cdots \star s_{j_{m+1}} \star s_{j_{m-1}} \max & \text{if} \quad m > 1; \\ \pi \ \text{ and } \ \rho & \text{if} \quad m = 1. \end{cases}$$

The latter case automatically proves our claim that the Bruhat cover between π and ρ is a reverse Bruhat cover. In the first case, it follows from our constructive Definition 5.7 that there is a reverse Bruhat cover between π and ρ as before. Therefore, our claim is proven.

To the converse statement we start with a reverse Bruhat order cover relation $\rho \lessdot \pi$ and apply the similar arguments as above. Hence our proof is complete.

□

Corollary 5.9. Let k be a positive integer and v be an element from W such that $\ell(v) = k$, $L(v \cdot \pi) = L(\pi) + k$, and $L(v \cdot \rho) = L(\rho) + k$. If $v \cdot \rho$ covers $v \cdot \pi$ in the Bruhat order, then ρ covers π in the Bruhat order.

Proof. Suppose $s_{i_m} \cdots s_{i_1}$ is a reduced expression for v. If $v \cdot \rho$ covers $v \cdot \pi$ in the Bruhat order, then there exists a chain of reverse Bruhat order covering relations $\pi_j \lessdot \rho_j$ for $j = 1, \ldots, m$, where π_{j-1} is obtained from π_j via $s_j \star \pi_j = \pi_{j-1}$, and similarly, ρ_{j-1} is obtained from ρ_j via $s_j \star \rho_j = \rho_{j-1}$. Note that $\pi_1 = \pi$ and $\rho_1 = \rho$.

It follows from Definition 5.7 that there are reverse Bruhat coverings between ρ_j and π_j for $j = 1, \ldots, m$. Now our claim follows from Theorem 5.8.

□

Lemma 5.10. Let π and ρ be two μ-involutions. Then π is covered by ρ in Bruhat order if and only if there exist a simple reflection s_j and elements $w_1, w_2 \in W$ such that $w_1 w_2 \in W^{-1}(\pi)$, $w_1 s_j w_2 \in W^{-1}(\rho)$, and

$$\ell(w_1 w_2) = \ell(w_1) + \ell(w_2).$$

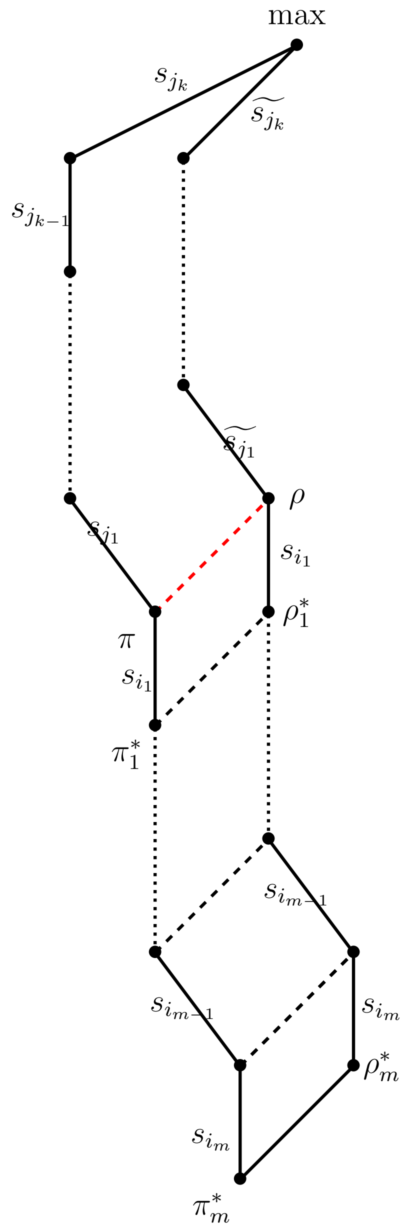

Figure 4: Depiction of a generic Bruhat order covering relation (in red). Thick, solid lines indicate weak order cover relations. The dashed lines are Bruhat order covers but they are not necessarily covers of the weak order.

Proof. Suppose $\pi \lessdot \rho$ is a covering relation. As in the proof of Theorem 5.8, by Timashev's characterization we know that there exists a simple reflection s_{i_1} and μ-involutions π_1^*, ρ_1^* such that π_1^* is covered by π in weak order (hence $s_{i_1} \cdot \pi_1^* = \pi$ and $\pi_1^* \neq \pi$), ρ_1^* is covered by ρ in weak order (hence $s_{i_1} \cdot \rho_1^* = \rho$ and $\rho_1^* \neq \rho$), and π_1^* is covered by ρ_1^* in Bruhat order. If the last covering relation is not a weak order cover, by Timashev's characterization once again, we know that there exists a simple reflection s_{i_2} ($i_1 \neq i_2$) and μ-involutions π_2^*, ρ_2^* such that $s_{i_2} \cdot \pi_2^* = \pi_1^*$, $s_{i_2} \cdot \rho_2^* = \rho_1^*$, and π_2^* is covered by ρ_2^* in Bruhat order. And this procedure continues until we reach at a weak order covering relation $\pi_m^* \lessdot \rho_m^*$ via $s_j \cdot \pi_m^* = \rho_m^*$ for some simple reflection s_j. For notation, we set $\pi_l^* = s_{i_{l+1}} \cdot \pi_{l+1}^*$, $\rho_l^* = s_{i_{l+1}} \cdot \rho_{l+1}^*$ for $l = 0, 1, \dots, m-1$. See Figure 5.2. Let $w_1 = s_{i_1} \cdots s_{i_m}$ and let w_2 denote any element from $W^{-1}(\pi_m^*)$. It is clear that $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$, $w_1 w_2 \in W^{-1}(\pi)$, and $w_1 s_j w_2 \in W^{-1}(\rho)$.

Conversely, if there exist $w_1, w_2 \in W$ such that $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$, $w_1 w_2 \in W^{-1}(\pi)$ and $w_1 s_j w_2 \in W^{-1}(\rho)$, then we set $\pi_m^* = w_2 \cdot \min$, $\rho_m^* = s_j \cdot w_2 \cdot \min$. Then π_m^* is covered by ρ_m^* in the weak order, or $\pi_m^* = \rho_m^*$. In the latter case, $\pi = \rho$ hence nothing to prove. In the former case, by the recursive definition of Bruhat order (Proposition 5.1) π is covered by ρ in Bruhat order.

□

Now we are ready to put together what we have and give a full description of the covering relations of Bruhat order.

Theorem 5.11. Let π be a μ-involution and ρ be a ν-involution. Then ρ covers π in Bruhat order if and only if one of the following holds:

1. μ is covered by ν in the refinement ordering and $W(\rho) \cap W(\pi) \neq \emptyset$.
2. $\nu = \mu$ and there exist a transposition t in the form $t = v s_j v^{-1}$, where s_j is a simple transposition and $v \in W$ such that
   (a) $L(\pi) - L(v \star \pi) = L(\rho) - L(v \star \rho) = \ell(v)$;
   (b) $tW^{-1}(\pi) \cap W^{-1}(\rho) \neq \emptyset$.

Proof. The first claim is the restatement of Theorem 5.6. The second claim follows from Lemma 5.10 if we set $v = w_1$. □

Example 5.12. Consider $\pi = [21|3]$. It is easily computed that $W^{-1}([21|3]) = \{312\}$. π covers two $(2,1)$-involutions $\rho_1 = [31|2]$ and $\rho_2 = [23|1]$. Note that $W^{-1}(\rho_1) = \{213\}$, $W^{-1}(\rho_2) = \{132\}$. For the covering $\rho_1 \lessdot \pi$, the transposition is $t = 132$ and for the covering $\rho_2 \lessdot \pi$, the transposition is $t = 321$. Finally, the W-set of π is $\{213\}$, and the only composition that is finer than $2+1$ is $1+1+1$. Among all $(1,1,1)$-involutions, the only degenerate involution whose W-set is a subset of $\{213\}$ is $\rho_3 = [2|1|3]$. Therefore, we found all degenerate involutions that are covered by π; these are ρ_1, ρ_2 and ρ_3.

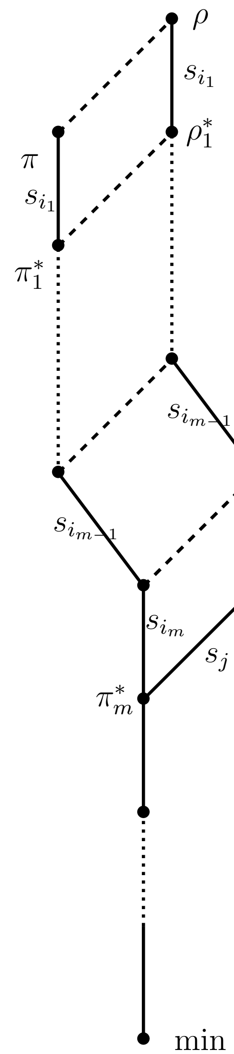

Figure 5: Bruhat order covering relations are determined by weak order coverings.

6 Cell decomposition and barred permutations

In this section, we construct a Białynicki-Birula cell decomposition of $\mathcal{X}_n$ and we construct two maps σ and τ that describe the structure of the cells.

Let $\lambda : \mathbb{C}^* \to T$ a one-parameter subgroup. Explicitly

$$\lambda(z) = \begin{pmatrix} z^{a_1} & 0 & \dots & 0 \\ 0 & z^{a_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & z^{a_n} \end{pmatrix},$$

where $z \in \mathbb{C}^*$ and $(a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$. We call a sequence $(a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$ of length n admissible if the following restrictions are satisfied

(a) $a_1 < a_2 < \cdots < a_n$;

(b) if $i, j, k < l$, then $a_i + a_j < a_k + a_l$;

(c) if $i, j < k$, then $2a_i < a_j + a_k$.

Lemma 6.1. For any $n > 0$, the sequence $(0, 1, 3, 7, \dots, 2^{n-1} - 1)$ is admissible.

Proof. Each of the three conditions for admissibility is easily checked by using binary representations of the a_i. □

A one-parameter subgroup corresponding to an admissible sequence is called an admissible one-parameter subgroup, and we fix λ corresponding to the admissible sequence given in Lemma 6.1. We first investigate the flow

$$\lim_{t \to 0} \lambda(t) \cdot Q_\pi$$

of a distinguished quadric Q_π associated to a μ-involution π under the action of λ.

Example 6.2. Let $\pi = (68)|(25)(4)(9)|(13)(7)$. Then Q_π consists of the flag $\mathcal{F} : 0 = V_0 \subset V_1 \subset V_2 \subset \mathbb{C}^9$ whose successive quotients are spanned by e_6, e_8, by e_2, e_4, e_5, e_9 and by e_1, e_3, e_7, as well the three quadric hypersurfaces defined on these successive quotients by $Q_1 = x_6x_8$, $Q_2 = x_2x_5 + x_4^2 + x_9^2$ and $Q_3 = x_1x_3 + x_7^2$. Since the quadric $Q_1 = x_6x_8$ is T-fixed, it is also λ-fixed. The quadric $Q_2 = x_2x_5 + x_4^2 + x_9^2$ is not λ-fixed and

$$\lambda(t) \cdot Q_2 = t^{-(a_2+a_5)}x_2x_5 + t^{-2a_4}x_4^2 + t^{-2a_9}x_9^2.$$

Since λ is admissible, $2a_9 > a_2 + a_5 > 2a_4$ and it follows that $\lim_{t\to 0} \lambda(t) \cdot Q_2$ is the sequence of quadrics x_4^2, x_2x_5, x_9^2. A similar calculation for $Q_3 = x_1x_3 + x_7^2$ yields

$$\lim_{t \to 0} Q_\pi = Q_{\pi'}$$

where $\pi' = (68)|(4)|(25)|(9)|(13)|(7)$ corresponding to the barred permutation [86|4|52|9|31|7].

In general, we define a map

$$\tau : \{\text{degenerate involutions}\} \to \{\text{barred permutations}\}$$

as follows. Suppose $\pi = [\pi_1|\pi_2|\ldots|\pi_k]$. For each π_j, order its cycles in lexicographic order on the largest value in each cycle. Then add bars between each cycle. Since π is a μ-involution, every cycle that occurs in each π_j has length one or two. Finally, convert one-cycles (i) into the numeral i and two-cycles (ij) with $i < j$ into the string ji. For example, $\tau((68)|(25)(4)(9)|(13)(7)) = [86|4|52|9|31|7]$.

Proposition 6.3. For any μ-involution π, $\lim_{t\to 0} \lambda(t) \cdot Q_\pi$ is the T-fixed quadric parameterized by $\tau(\pi)$.

Proof. The λ-weight of a monomial $x_i x_j$ is $-(a_i + a_j)$. (This includes the case when $i = j$.) By the definition of admissibility, the monomial $x_i x_j$ has a smaller λ-weight than $x_k x_l$ if and only if $\max(i, j) < \max(k, l)$. This implies that the distinguished complete quadric Q_π flows under the λ action to the distinguished T-fixed complete quadric associated to $\tau(\pi)$. □

Next, we define a map in the opposite direction

$$\sigma : \{\text{barred permutations}\} \to \{\text{degenerate involutions}\}$$

which has the following geometric interpretation. Let Q_α be the T-fixed quadric associated to a barred permutation α. Then $\sigma(\alpha)$ corresponds to the distinguished quadric in the dense B-orbit of the cell that contains Q_α. In other words, the B-orbit of $Q_{\sigma(\alpha)}$ has the largest dimension among all B-orbits that flow to Q_α.

Definition 6.4. Let $\alpha = [\alpha_1|\alpha_2|\ldots|\alpha_k]$ be a barred permutation. Let d_j denote the largest value occurring in α_j, giving rise to a sequence $\mathbf{d} = (d_1, d_2, \ldots, d_k)$. For example, if $\alpha = [86|9|52|4|7|31]$, then $\mathbf{d} = (8, 9, 5, 4, 7, 3)$. We say that π has a *descent* (resp., *ascent*) at position i if $\mathbf{d}$ has a descent (resp., ascent) at position i.

The μ-involution $\sigma(\alpha)$ is constructed by first converting strings i of length 1 into one-cycles (i) and strings ji of length 2 into two-cycles (ij). Then remove the bars at positions of ascent and keep the bars at positions of descent in α. For example, $\sigma([86|4|52|9|31|7]) = (68)|(25)(4)(9)|(13)(7)$.

Proposition 6.5. For any barred permutation α, the B-orbit of $Q_{\sigma(\alpha)}$ has the largest dimension among all B-orbits that flow to Q_α.

Proof. It follow immediately from Proposition 6.3 that $\lim_{t\to 0} \lambda(t) \cdot Q_{\sigma(\alpha)} = Q_\alpha$. Let μ be the composition associated to $\sigma(\alpha)$. If $\lim_{t\to 0} \lambda(t) \cdot Q_\pi = Q_\alpha$ for a ν-involution π, then ν must be a refinement of μ. For at each position of ascent in the sequence $\mathbf{d}$, if a bar does not exist at that position in π, Proposition 6.3 implies that the flow of Q_π would not be toward Q_α. By [7], the dense B-orbit of the cell is the one that intersects the largest dimensional G-orbit, so the dense B-orbit is indeed the one containing $Q_{\sigma(\alpha)}$. □

Given a barred permutation α, let $w(\alpha)$ denote the permutation in one-line notation that is obtained by removing all bars in α. Let $\mathrm{inv}(\alpha)$ denote the number of length 2 strings that occur and let $\mathrm{asc}(\alpha)$ denote the number of ascents in α.

Lemma 6.6. The dimension of the cell containing the T-fixed quadric parameterized by α is $\ell(w_0) - \ell(w(\alpha)) + \mathrm{inv}(\alpha) + \mathrm{asc}(\alpha)$.

Proof. Since $w(\alpha)$ belongs to the $\mathcal{W}$-set of the B-orbit containing Q_α, the codimension of the B-orbit containing Q_α in its G-orbit is $\ell(w(\alpha))$ [10]. As the codimension of the closed G-orbit in the G-orbit containing Q_α is $\mathrm{inv}(\alpha)$ and the dimension of the closed G-orbit is $\ell(w_0)$, it follows that the dimension of the B-orbit containing Q_α is $\ell(w_0)+\mathrm{inv}(\alpha)-\ell(w(\alpha))$. The result follow from the further observation that the codimension of the B-orbit containing Q_α in its cell is $\mathrm{asc}(\alpha)$. □

In Figure 6, we depict the cell decomposition of $\mathcal{X}_3$, the variety of complete conics in $\mathbb{P}^2$. Each colored rectangle represents a B-orbit parameterized by its corresponding μ-involution, and the edges stand for the covering relations in Bruhat order. A cell is a union of all B-orbits of the same color.

Given a Białynicki-Birula cell decomposition, it is not always true that the closure of a cell is the union of other cells. In other words, such a decomposition is not always a stratification. However, it is still a very interesting question to study the partial order on cells (or on the indexing barred permutations) defined by

$$\alpha \leq \beta \iff C_\alpha \leq C_\beta \iff C_\alpha \subseteq \overline{C_\beta},$$

where α, β are barred permutations of length n, and C_α, C_β denote the Białynicki-Birula cells with fixed points Q_α, Q_β. We illustrate the resulting cell decomposition when $n = 3$ in Figure 7. The dimension of a cell corresponding to a vertex in the figure is equal to the length of any chain from the bottom cell. A vertex corresponding to cell C is connected by an edge to a vertex of a cell C' of dimension one lower if and only if $C' \subseteq \overline{C}$. (The order complex of such a poset is considered in the interesting paper [16] of Knutson.)

Remark 6.7. For $n \geq 3$, the Białynicki-Birula decomposition of $\mathcal{X}_n$ is not a stratification. To see this, we consider the Bruhat order on $\mathcal{X}_3$, depicted in Figure 6. The closure of the pink cell $C_{[1|3|2]}$ intersects the orange cell $C_{[3|1|2]}$ in $\mathcal{O}^{[3|1|2]}$, which is non-empty, but not equal to the entire orange cell which also includes $\mathcal{O}^{[3|12]}$.

It is desirable to have a combinatorial rule determining the (covering) relations of Bruhat order which does not go through the costly inductive procedure given in Section 5. Given a composition μ of n, let us denote by $\mathcal{B}_{Cell}(\mu)$ the set of all μ-involutions such that $\mathcal{O}^\mu$ is the dense B-orbit of its cell. Experimentally, we have observed that the inclusion order restricted to $\mathcal{B}_{Cell}(\mu)$ is a ranked poset with a minimal and a maximal element. In Figure 8 we depict $B(1,3)$ as an embedded subposet in the closure order on $(1,3)$-involutions. We conclude by posing this observation as a conjecture.

Conjecture 6.8. Let $\mathcal{B}_{Cell}(\mathcal{X})$ denote the set of all B-orbits in $\mathcal{X}$ that are dense in their Białynicki-Birula cells. The restriction of Bruhat order to $\mathcal{B}_{Cell}(X)$ is a graded poset with a maximum and a minimum element.

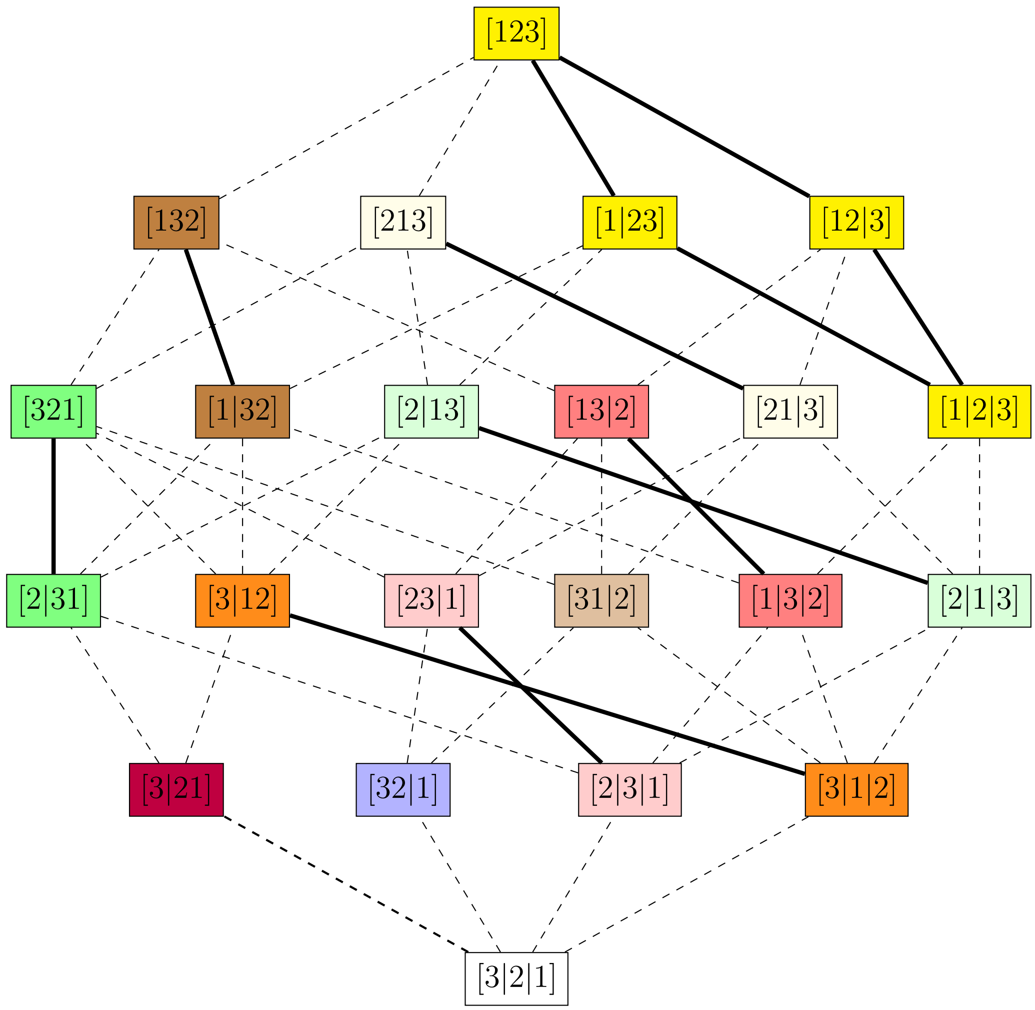

Figure 6: Cell decomposition and the Bruhat order for $\mathcal{X}_3$.

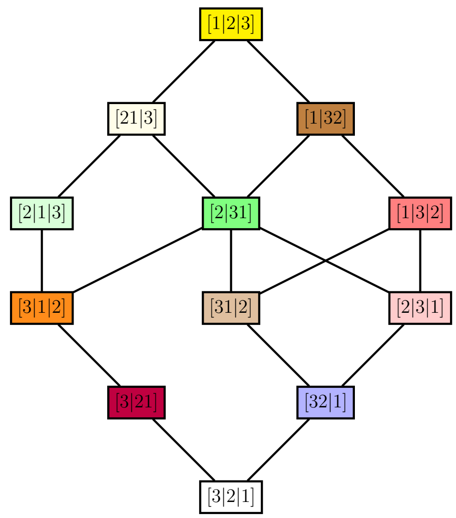

Figure 7: Cell closure inclusion poset for $\mathcal{X}_3$. The labels give the barred permutation parametrizing the T-fixed point in the cell.

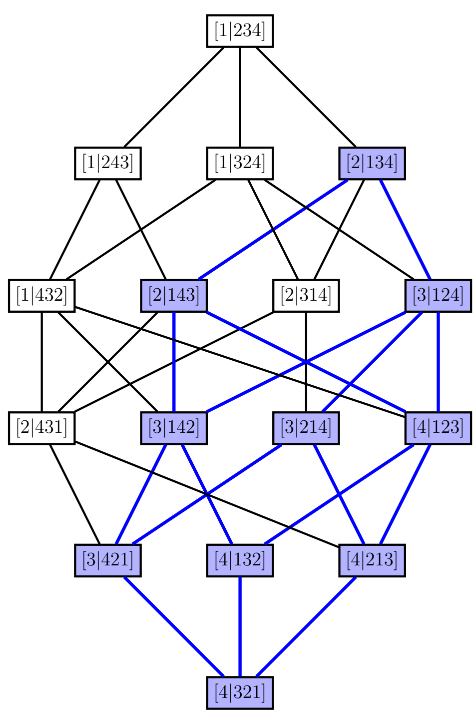

Figure 8: The dense B-orbits of BB-cells contained in $(1,3)$-involutions wrt closure order.

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