SOME PLETHYSTIC IDENTITIES AND KOSTKA-FOULKES POLYNOMIALS.

MAHIR BILEN CAN

1. Introduction.

Symmetric functions \( \{ E_{n,k}(X) \}_{k=1}^{n} \), defined by the Newton interpolation
\[
e_n[X \frac{1-z}{1-q}] = \sum_{k=1}^{n} (z;q)_k \frac{E_{n,k}(X)}{(q;q)_k}
\]
plays an important role in the Garsia-Haglund proof of the \( q,t \)-Catalan conjecture, [2].

Let \( \Lambda^n_{Q(q,t)} \) be the space of symmetric functions of degree \( n \), over the field of rational functions \( Q(q,t) \), and let \( \nabla : \Lambda^n_{Q(q,t)} \rightarrow \Lambda^n_{Q(q,t)} \) be the Garsia-Bergeron operator.

By studying recursions, Garsia and Haglund show that the coefficient of the elementary symmetric function \( e_n(X) \) in the image \( \nabla(E_{n,k}(X)) \) of \( E_{n,k}(X) \) is equal to the following combinatorial summation
\[
(1.1) \quad \langle \nabla(E_{n,k}(X)), e_n(X) \rangle = \sum_{\pi \in D_{n,k}} q^{\text{area}(\pi)} t^{\text{bounce}(\pi)},
\]
where \( D_{n,k} \) is the set of all Dyck paths with initial \( k \) North steps followed by an East step. Here \( \text{area}(\pi) \) and \( \text{bounce}(\pi) \) are two numbers associated with a Dyck path \( \pi \). It is conjectured in [4], more generally, that the \( \nabla E_{n,k}(X) \) are “Schur positive.”

In [1], using (1.1), Can and Loehr prove the \( q,t \)-Square conjecture of the Loehr and Warrington [7].

The aim of this article is to understand the functions \( \{ E_{n,k}(X) \}_{k=1}^{n} \) better. We prove that the vector subspace generated by the set \( \{ E_{n,k}(X) \}_{k=1}^{n} \) of the space \( \Lambda^n_{Q(q)} \) of degree \( n \) symmetric functions over the field
\[ (\mathbb{Q}(q), \text{equal to the subspace generated by} \\ \{ s_{(k,1^{n-k})}[X/(1-q)] \}_{k=1}^{n}, \]

Schur functions of hook shape, plethystically evaluated at \( X/(1-q) \).

In particular, we determine explicitly the transition matrix and its inverse from \( \{ E_{n,k}(X) \}_{k=1}^{n} \) to \( \{ s_{(k,1^{n-k})}[X/(1-q)] \}_{k=1}^{n} \). The entries of the matrix turns out to be cocharge Kostka-Foulkes polynomials.

We find the expansion of \( E_{n,k}(X) \) into the Hall-Littlewood basis, and as a corollary we recover a closed formula for the cocharge Kostka-Foulkes polynomials \( \tilde{K}_{\lambda,\mu}(q) \) when \( \lambda \) is a hook shape;

\[
\tilde{K}_{(n-k,1^k)}(q) = (-1)^k \sum_{i=0}^{k} (-1)^i q^{i\binom{r}{i}}.
\]

Here, \( \mu \) is a partition of \( n \) whose first column is of height \( r \).

2. **Background.**

2.0.1. **Notation.** A partition \( \mu \) of \( n \in \mathbb{Z}_{\geq 0} \), denoted \( \mu \vdash n \), is a nonincreasing sequence \( \mu_1 \geq \mu_2 \geq \ldots \geq \mu_k > 0 \) of numbers such that \( \sum \mu_i = n \). The conjugate partition \( \mu' = \mu'_1 \geq \ldots \geq \mu'_s > 0 \) is defined by setting \( \mu'_i = |\{ \mu_r : \mu_r \geq i \}| \).

\( \text{Par}(n,r) \) denotes the set of all partitions \( \mu \vdash n \) whose biggest part is equal to \( \mu_1 = r \).

We identify a partition \( \mu \) with its Ferrers diagram, in French notation. Thus, if the parts of \( \mu \) are \( \mu_1 \geq \mu_2 \geq \ldots \geq \mu_k > 0 \), then the corresponding Ferrers diagram have \( \mu_i \) lattice cells in the \( i^{th} \) row (counting from bottom to up).

Following Macdonald, [8] the arm, leg, coarm and coleg of a lattice square \( s \) are the parameters \( a_\mu(s), l_\mu(s), a'_\mu(s) \) and \( l'_\mu(s) \) giving the number of cells of \( \mu \) that are respectively strictly EAST, NORTH, WEST and SOUTH of \( s \) in \( \mu \).

Given a partition \( \mu = (\mu_1, \mu_2, \ldots, \mu_k) \), we set

\[
n(\mu) = \sum_{i=1}^{k} (i-1)\mu_i = \sum_{s \in \mu} l_\mu(s).
\]
We also set
\[
\tilde{h}_\mu(q, t) = \prod_{s \in \mu} (q^{a_s} - t^{a_s} + 1) \quad \text{and} \quad \tilde{h}_\mu'(q, t) = \prod_{s \in \mu} (t^{a_s} - q^{a_s} + 1).
\]

Let $F$ be a field, and let $X = \{x_1, x_2, \ldots\}$ be an alphabet (a set of indeterminates). The algebra of symmetric functions over $F$ with the variable set $X$ is denoted by $\Lambda_F(X)$.

If $Q \subseteq F$, it is well known that $\Lambda_F(X)$ is freely generated by the set of power-sum symmetric functions $\{p_r(X) : r = 1, 2, \ldots\}$ and $p_r(X) = x_1^r + x_2^r + \cdots$.

The algebra, $\Lambda_F(X)$ has a natural grading (by degree).

\[
\Lambda_F(X) = \bigoplus_{n \geq 0} \Lambda^n_F(X),
\]

where $\Lambda^n_F(X)$ is the space of homogenous symmetric functions of degree $n$.

A basis for the vector space $\Lambda^n_F(X)$ is given by the set $\{p_\mu(X)\}_{\mu \vdash n}$,

\[
p_\mu(X) = \prod_{i=1}^k p_{\mu_i}(X), \quad \text{where} \quad \mu = \sum_{i=1}^k \mu_i.
\]

Another basis for $\Lambda^n_F(X)$ is given by the Schur functions $\{s_\mu(X)\}_{\mu \vdash n}$, where $s_\mu(X)$ is defined as follows. Let

\[
e_n(X) = \sum_{1 \leq i_1 < \cdots < i_n} x_{i_1}x_{i_2} \cdots x_{i_n}
\]

be the $n$’th elementary symmetric function. If $\mu = \sum_{i=1}^k \mu_i$, then

\[
s_\mu(X) = \det(e_{\mu_i-i+j}(X))_{1 \leq i, j \leq m},
\]

where $\mu'_i$ is the $i$’th part of the conjugate partition $\mu' = (\mu'_1, \ldots, \mu'_l)$ and $m \geq l$.

2.0.2. **Plethysm.** For the purposes of this section, we represent an alphabet $X = \{x_1, x_2, \ldots\}$ as a formal sum $X = \sum x_i$. Thus, if $Y = \sum y_i$ is another alphabet, then

\[
XY = (\sum x_i)(\sum y_i) = \sum_{i,j} x_iy_j = \{x_iy_j\}_{i,j \geq 1},
\]
and

\[(2.7) \quad X + Y = \left( \sum_i x_i \right) + \left( \sum_j y_j \right) = \{x_i, y_j\}_{i,j \geq 1}.\]

The formal additive inverse, denoted \(-X\), of an alphabet \(X = \sum x_i\) is defined so that \(-X + X = 0\).

In this vein, if \(p_k(X) = \sum_{k \geq 1} x^k\) is a power sum symmetric function, we define

\[(2.8) \quad p_k[XY] = p_k[X]p_k[Y],\]
\[(2.9) \quad p_k[X + Y] = p_k[X] + p_k[Y],\]
\[(2.10) \quad p_k[-X] = -p_k[X].\]

This operation is called plethysm. Since \(\Lambda_X\) is freely generated by the power sums, the plethysm operator can be extended to the other symmetric functions. In fact, using plethysm, one defines the following bases for \(\Lambda^n_{\mathbb{Q}(q)}\) and \(\Lambda^n_{\mathbb{Q}(q,t)}\), respectively.

**Theorem-Definition 1.** (cocharge Hall-Littlewood polynomials)

There exists a basis \(\{\tilde{H}_\mu(X; q)\}_{\mu \vdash n}\) for the vector space \(\Lambda^n_{\mathbb{Q}(q)}\), which is uniquely characterized by the properties

1. \(\tilde{H}_\mu(X; q) \in \mathbb{Z}[q]\{s_\lambda : \lambda \geq \mu\}\),
2. \(\tilde{H}_\mu[(1-q)X; q] \in \mathbb{Z}[q]\{s_\lambda : \lambda \geq \mu'\}\),
3. \(\langle \tilde{H}_\mu(X; q), s_{(n)} \rangle = 1\).

**Theorem-Definition 2.** (Modified Macdonald polynomials)

There exists a basis \(\{\tilde{H}_\mu(X; q, t)\}_{\mu \vdash n}\) for the vector space \(\Lambda^n_{\mathbb{Q}(q,t)}\), which is uniquely characterized by the properties

1. \(\tilde{H}_\mu(X; q, t) \in \mathbb{Z}[q, t]\{s_\lambda : \lambda \geq \mu\}\),
2. \(\tilde{H}_\mu[(1-q)(1-t)X; q, t] \in \mathbb{Z}[q, t]\{s_\lambda : \lambda \geq \mu'\}\),
3. \(\langle \tilde{H}_\mu(X(1-t); q, t), s_{(n)} \rangle = 1\).

It follows from these Theorem-Definitions that

\[(2.11) \quad \tilde{H}_\mu(X; 0, t) = \tilde{H}_\mu(X; t),\]
\[(2.12) \quad \tilde{H}_\mu(X; q, t) = \tilde{H}_{\mu'}(X; t, q).\]
2.0.3. Kostka-Foulkes and Kostka-Macdonald polynomials. Let \( \widetilde{H}_\mu(X; q) = \sum \widetilde{K}_{\lambda \mu}(q)s_\lambda \) and \( \widetilde{H}_\mu(X; q, t) = \sum \widetilde{K}_{\lambda \mu}(q, t)s_\lambda \) be, respectively, the Schur basis expansions of the Hall-Littlewood and Macdonald symmetric functions. The coefficients of the Schur functions are called, respectively, the cocharge Kostka-Foulkes polynomials, and the modified Kostka-Macdonald polynomials. It is known that \( \widetilde{K}_{\lambda \mu}(q, t), \widetilde{K}_{\lambda \mu}(q) \in \mathbb{N}[q, t] \).

It follows from equation (2.11) and the Schur basis expansions that

\[
(2.13) \quad \widetilde{K}_{\lambda \mu}(0, t) = \widetilde{K}_{\lambda \mu}(t).
\]

2.0.4. Cauchy Identities. Let \( X = \sum x_i \) be an alphabet, and let \( \Omega[X] = \exp(\sum_{k=1}^{\infty} p_k(X)/k) \). Then,

\[
(2.14) \quad \Omega[X] = \prod_i \frac{1}{1-x_i} = \sum_{n=0}^{\infty} s_n(X),
\]

\[
(2.15) \quad \Omega[X] = \prod_i (1-x_i) = \sum_{n=0}^{\infty} s_{1n}(X).
\]

If \( Y = \sum y_i \) is another alphabet, then

\[
(2.16) \quad e_n[XY] = \sum_{\mu \vdash n} s_\mu[X]s_{\mu'}[Y],
\]

\[
(2.17) \quad e_n[XY] = \sum_{\mu \vdash n} \frac{\widetilde{H}_\mu[X; q, t]\tilde{H}_\mu[Y; q, t]}{\tilde{h}_\mu(q, t)\tilde{h}'_\mu(q, t)},
\]

where \( \tilde{h}_\mu(q, t) \) and \( \tilde{h}'_\mu(q, t) \) are as in (2.2).

\[
(2.18) \quad s_\mu[1-z] = \begin{cases} (-z)^k(1-z) & \text{if } \mu = (n-k, 1^k), \\ 0 & \text{otherwise.} \end{cases}
\]

2.0.5. Cauchy’s q-binomial theorem. Let \( (z; q)_k = (1-z)(1-qz)\cdots(1-q^{k-1}z) \), and let

\[
\binom{k}{r} = \frac{(q; q)_k}{(q; q)_r(q; q)_{k-r}}.
\]
Then, the Cauchy \(q\)-binomial theorem states that

\[
(z; q)_k = \sum_{r=0}^{k} z^r (-1)^r e_r[1, q, \ldots, q^{k-1}] = \sum_{r=0}^{k} z^r q^{(i)} (-1)^r \left\lceil \frac{k}{r} \right\rceil.
\]

3. **Symmetric Functions** \(E_{n,k}(X)\).

The family \(\{E_{n,k}(X)\}_{k=1}^{n}\) of symmetric functions are defined by the plethystic identity

\[
e_n\left[X\frac{1-z}{1-q}\right] = \sum_{k=1}^{n} \frac{(z; q)_k}{(q; q)_k} E_{n,k}(X).
\]

Let \(0 \leq k \leq r\), and let

\[
T_{k+1,r} = (-1)^k \sum_{i=0}^{k} (-1)^i q^{(i)} \left\lceil \frac{r}{i} \right\rceil.
\]

**Proposition 3.1.** For \(k = 0, \ldots, n - 1\),

\[
s_{(k+1,1^{n-k-1})}\left[X/(1-q)\right] = \sum_{r=k+1}^{n} T_{k+1,r} \frac{E_{n,r}(X)}{(q; q)_r}.
\]

**Proof.** Using the Cauchy \(q\)-binomial theorem, we see that the coefficient of \((-z)^k\) on the right hand side of (3.1) is

\[
q^{(b)} \sum_{i=0}^{n-k} \left[ \begin{array}{c} k + i \ \\
\ \\
\ \\
k \end{array} \right] \frac{E_{n,k+i}}{(q; q)_{k+i}}.
\]
On the other hand, by the identities (2.16) and (2.18),

\[ e_n[X \frac{1-z}{1-q}] = \sum_{\lambda} s_{\lambda}^2 X^{\lambda} s_{\lambda'} \]

\[ = \sum_{\lambda'=(n-r,1^r)} X^\lambda (-z)^r (1-z) \]

\[ = \sum_{r=0}^{n-1} s_{(r+1,1^{n-r-1})} X^\lambda (-z)^r (1-z) \]

\[ = s_{1^n} \left( X^{\lambda } \right) + \cdots + (-z)^n s_{n} \left( X^{\lambda } \right). \]

Comparing the coefficient of \((-z)^k\) gives, for \(k \geq 1\),

(3.5)

\[ q^{(k)} \sum_{i=0}^{n-k} \binom{k+i}{k} \frac{E_{n,k+i}}{q(q;q)_{k+i}} = s_{k,1^{n-k}} \left( X^{\lambda } \right) + s_{k+1,1^{n-k-1}} \left( X^{\lambda } \right), \]

and

(3.6)

\[ \sum_{i=1}^{n} \frac{E_{n,i}}{(q;q)_i} = s_{1^n} \left( X^{\lambda } \right). \]

We take the alternating sums of the equations (3.5) and (3.6) to get

\[ s_{k+1,1^{n-k-1}} \left[ X/(1-q) \right] = (-1)^k \left( \sum_{j=1}^{n} \frac{E_{n,j}}{(q;q)_j} \right) + \sum_{i=1}^{k} (-1)^{k+i} q^{(i)} \sum_{j=0}^{n-i} \binom{i+j}{i} \frac{E_{n,i+j}}{(q;q)_{i+j}}. \]

By collecting \(E_{n,k}(X)\)'s, and using (2.19) we obtain

\[ s_{(k+1,1^{n-k-1})} \left[ X/(1-q) \right] = \sum_{r=k+1}^{n} T_{k+1,r} \frac{E_{n,r}(X)}{(q;q)_r}. \]

\[ \square \]
Let $S$ and $E$ be the matrices
\[
S = \begin{pmatrix}
\frac{X}{1-q}
\end{pmatrix}
\begin{bmatrix}
\frac{s_1}{(1-q)}
& \frac{s_2}{(1-q)}
& \vdots
& \frac{s_n}{(1-q)}
\end{bmatrix}
\quad \text{and} \quad
E = \begin{pmatrix}
\frac{E_0}{(q;q)_1}
& \frac{E_1}{(q;q)_2}
& \vdots
& \frac{E_n}{(q;q)_n}
\end{pmatrix},
\]
respectively, and let $T$ be the transition matrix from $E$ to $S$, so that $S = TE$. Then, $T$ is an upper triangular matrix with the $k+1, r$’th entry
\[
T_{k+1, r} = (-1)^k \sum_{i=0}^{k} (-1)^i q^{r(i)} \left( \begin{array}{c}
\left[ \frac{r}{i} \right]
\end{array} \right).
\]
For example, when $n = 5$,
\[
T = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & q & (q + 1) & q^2 + q + 1 & q^3 + q^2 + q + 1 \\
0 & 0 & q^3 & q^3(q + 1) & q^3(q^2 + 2q^2 + q + 1) \\
0 & 0 & 0 & q^6 & q^6(q^3 + q^2 + q + 1) \\
0 & 0 & 0 & 0 & q^{10}
\end{pmatrix}.
\]
Then,
\[
T^{-1} = \begin{pmatrix}
1 & -q^{-1} & q^{-2} & -q^{-3} & \ldots \frac{q^{-4}}{q^4} \\
0 & q^{-1} & -\frac{q^2 + q + 1}{q^2} & \frac{q^4 + 2q^2 + q + 1}{q^4} \\
0 & 0 & q^{-3} & -\frac{q^4 + 2q^2 + q + 1}{q^4} & \frac{q^5 + 3q^3 + 2q^2 + q + 1}{q^5} \\
0 & 0 & 0 & q^{-6} & \frac{q^5 + 3q^3 + 2q^2 + q + 1}{q^5} \\
0 & 0 & 0 & 0 & q^{-10}
\end{pmatrix}.
\]

**Proposition 3.2.** $T^{-1}$ is (necessarily) upper triangular, and its $k+1, r$’th entry is equal to
\[
(T^{-1})_{k+1, r} = (-1)^r q^{r(k+1)} T_{k+1, r}.
\]

**Proof.** Let $L$ be the upper triangular matrix with the $k + 1, r$’th entry
\[
L_{k+1, r} = (-1)^r q^{r(k+1)} T_{k+1, r}
\]
for $r > k$. 

Clearly, $TL$ is an upper triangular matrix, and the $i+1$, $j$’th entry of $TL$ is

\begin{equation}
(TL)_{i+1,j} = \sum_{k=1}^{n} T_{i+1,k} L_{k,j}.
\end{equation}

It is straightforward to check that $(TL)_{i+1,i+1} = 1$. We use induction on $j$ to prove that for all $i + 1 < j$, $(TL)_{i+1,j} = 0$. So, we assume that for all $i + 1 < j$, $(TL)_{i+1,j} = 0$, and we are going prove that for all $i + 1 < j + 1$, $(TL)_{i+1,j+1} = 0$.

First of all, using the $q$-binomial identity

\begin{equation}
\begin{bmatrix} r \\ m \end{bmatrix} = \begin{bmatrix} r-1 \\ m \end{bmatrix} + \begin{bmatrix} r-1 \\ m-1 \end{bmatrix} q^{r-m}, \text{ for } m \geq 0,
\end{equation}

it is easy to show that

\begin{equation}
T_{i+1,k} = T_{i+1,k-1} + q^iT_{i,k-1}.
\end{equation}

It follows that

\begin{equation}
L_{k+1,j+1} = -q^{-(k+1)}L_{k+1,j} + q^{-k}L_{k,j}.
\end{equation}

Therefore,

\begin{align*}
\sum_{k=i+1}^{j+1} T_{i+1,k} L_{k,j+1} &= \sum_{k=i+1}^{j+1} T_{i+1,k} (-q^{-(k+1)}L_{k,j} + q^{-k}L_{k-1,j-1}) \\
&= \sum_{k=i+1}^{j+1} -q^{-k}T_{i+1,k}L_{k,j} + \sum_{k=i+1}^{j+1} q^{-(k-1)}T_{i+1,k}L_{k-1,j}.
\end{align*}

Using (3.10) in the last summation, we have

\begin{align*}
\sum_{k=i+1}^{j+1} T_{i+1,k} L_{k,j+1} &= \sum_{k=i+1}^{j+1} -q^{-k}T_{i+1,k}L_{k,j} + \sum_{k=i+1}^{j+1} q^{-(k-1)}T_{i+1,k-1}L_{k-1,j} \\
&+ \sum_{k=i+1}^{j+1} qT_{i+1,k-1}L_{k-1,j}.
\end{align*}
After rearranging the indices, and using the induction hypotheses, the right hand side of the equation simplifies to 0. Therefore, the proof is complete.

\[ \square \]

**Corollary 3.3.** Let \( A \subseteq \Lambda^n_{Q(q)}(X) \) be the \( n \)-dimensional subspace generated by the set \( \{ E_{n,k}(X) \}_{k=1}^n \), and let \( B \subseteq \Lambda^n_{Q(q)}(X) \) be the \( n \)-dimensional subspace generated by \( \{ s_{k,1^{n-k}}[X/(1-q)] \}_{k=1}^n \). Then, \( A = B \).

**Proof.** It is clear by Proposition 3.2 that \( A = B \). The dimension claim follows from Proposition 4.1 below.

\[ \square \]

The expression \((-1)^n p_n = \sum_{k=0}^{n-1} (-1)^k s_{k+1,1^{n-k-1}} \) is the bridge between Schur functions of hook type with the power sum symmetric functions. By the linearity of plethysm we have

\[
(-1)^n p_n[X]/(1-q^n) = (-1)^n p_n[X/(1-q)] = \sum_{k=0}^{n-1} (-1)^k s_{k+1,1^{n-k-1}}[X/(1-q)],
\]

and therefore

\[ (3.12) \quad (-1)^n p_n = (1 - q^n) \sum_{k=0}^{n-1} (-1)^k s_{k+1,1^{n-k-1}}[X/(1 - q)]. \]

**Corollary 3.4.** For all \( n \geq 1 \),

\[ (3.13) \quad (-1)^n p_n = \sum_{r=1}^{n} \frac{1 - q^n}{1 - q^r} E_{n,r}. \]

**Proof.** By Proposition 3.1 and (3.12) we get

\[ (3.14) \quad (-1)^n p_n = (1 - q^n) \sum_{k=0}^{n-1} \sum_{r=k+1}^{n} (-1)^k T_{k+1,r} \frac{E_{n,r}(X)}{(q; q)_r}. \]

By rearranging the summations and using the Cauchy’s \( q \)-binomial theorem once more, we finish the proof.

**Proposition 4.1.** For $k = 1, \ldots, n$,

\[
\frac{E_{n,k}(X)}{(q; q)_k} = \sum_{\mu \in \text{Par}(n,k)} \frac{\widetilde{H}_{\mu'}(X; q)}{h_{\mu}(q, 0) h'_{\mu}(q, 0)} = \sum_{\mu \in \text{Par}(n,k)} \frac{(-q)^n q^{2n(\mu')}}{\prod_{s \in \mu, l_s(\mu) = 0} (1 - q^{-a_s(\mu)-1})}.
\]

**Proof.** Let $Y = (1 - t)(1 - z)$. Then, by the Cauchy identity (2.17), we have

\[
\sum_{k=1}^n (z; q)_k E_{n,k}(X) = \sum_{\mu \vdash n} \frac{\widetilde{H}_{\mu'}[X; q, t] \widetilde{H}_{\mu}[(1 - t)(1 - z); q, t]}{h_{\mu}(q, t) h'_{\mu}(q, t)}.
\]

The left hand side of the equation (4.1) is independent of the variable $t$. Since $\widetilde{h}_{\mu}(q, 0) \neq 0$, and since $\widetilde{h}'_{\mu}(q, 0) \neq 0$, we are allowed to make the substitution $t = 0$ on both sides of the equation.

Note that

\[
\widetilde{h}_{\mu}(q, 0) \widetilde{h}'_{\mu}(q, 0) = \prod_{s \in \mu} q^{a_s(\mu)} \prod_{s \in \mu, l_s(\mu) \neq 0} (-q^{a_s(\mu)} + 1) \prod_{s \in \mu, l_s(\mu) = 0} (1 - q^{a_s(\mu) + 1})
\]

\[
= (-q)^n \prod_{s \in \mu} q^{2a_s(\mu)} \prod_{s \in \mu, l_s(\mu) \neq 0} \frac{1 - q^{a_s(\mu) + 1}}{-q^{a_s(\mu) + 1}}
\]

\[
= (-q)^n \prod_{s \in \mu} q^{2a_s(\mu)} \prod_{s \in \mu, l_s(\mu) = 0} (1 - q^{-a_s(\mu) - 1})
\]

\[
= (-q)^n q^{2n(\mu')} \prod_{s \in \mu, l_s(\mu) = 0} (1 - q^{-a_s(\mu) - 1}).
\]

The equality (4.5) follows from (2.1).

Using the Schur expansion $\widetilde{H}_{\mu}(X; q, t) = \sum_{\lambda} \widetilde{K}_{\lambda\mu}(q, t) s_{\lambda}$, we see that the plethystic substitution $X \rightarrow (1 - z)$, followed by the evaluation at $t = 0$ is the same as the evaluation $\widetilde{H}_{\mu}(X, q, 0)$ at $t = 0$, followed by the plethystic substitution $X \rightarrow (1 - z)$. Also, by Corollary 3.5.20 of [6], we know that

\[
\widetilde{H}_{\mu}[1 - z; q, t] = \Omega[-z B_{\mu}], \text{ where } B_{\mu} = \sum_{i \geq 1} t^{i-1} \frac{1 - q^{\mu_i}}{1 - q}.
\]
Therefore,

\[(4.6) \quad \tilde{H}_\mu[z, q, 0] = \Omega[-zB_\mu]_{t=0} \]
\[(4.7) \quad = \Omega[-z(1 + q + \cdots + q^{\mu_1-1})] \]
\[(4.8) \quad = \prod_{i=0}^{\mu_1-1} (1 - zq^i) \]
\[(4.9) \quad = (z; q)_{\mu_1}. \]

It follows from (2.11) and (2.12) that

\[(4.10) \quad \tilde{H}_\mu(X; q, 0) = \tilde{H}_\mu'(X; q). \]

By combining (4.1), (4.5), (4.9) and (4.10), we get

\[(4.11) \quad \sum_{k=1}^{n} (z; q)_k E_{n,k}(X) (q; q)_k = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu'(X; q)}{(q; q)_k} \frac{\tilde{H}_\mu'(X; q)}{\prod_{s \in \mu, l(s) = 0} (1 - q^{-a_s(s)-1})}. \]

By comparing the coefficient of \((z; q)_k\) in (4.11), we find that

\[(4.12) \quad E_{n,k}(X) = \sum_{\mu \vdash n} \frac{\tilde{K}_{\lambda', \mu'}(q, 0) \tilde{H}_\mu'(X; q)}{h_\mu(q, 0) h_\mu'(q, 0)} \frac{\tilde{H}_\mu'(X; q)}{\prod_{s \in \mu, l(s) = 0} (1 - q^{-a_s(s)-1})}. \]

Hence, the proof is complete.

\[\square\]

**Lemma 4.2.** Let \(\lambda \vdash n\) be a partition of \(n\). Then,

\[(4.13) \quad s_{\lambda}[X] \frac{1}{(1 - q)(1 - t)} = \sum_{\mu \vdash n} \frac{\tilde{K}_{\lambda', \mu'}(q, t) \tilde{H}_\mu(X; q, t)}{h_\mu(q, 0) h_\mu'(q, 0)}. \]

**Proof.** This follows from Theorem 1.3 of [3]. \(\square\)

**Corollary 4.3.** Let \(\lambda \vdash n\) be a partition. Then,

\[(4.14) \quad s_{\lambda}[X] \frac{1}{1 - q} = \sum_{\mu} \frac{\tilde{K}_{\lambda', \mu'}(q) \tilde{H}_\mu'(X; q)}{h_\mu(q, 0) h_\mu'(q, 0)} = \sum_{\mu} \frac{\tilde{K}_{\lambda', \mu'}(q) \tilde{H}_\mu'(X; q)}{(-q)^n q^{2n(\mu')} \prod_{s \in \mu, l(s) = 0} (1 - q^{-a_s(s)-1})}. \]

**Proof.** It follows from (2.11) and (2.12) that \(\tilde{K}_{\lambda', \mu'}(q, 0) = \tilde{K}_{\lambda', \mu'}(0, q) = \tilde{K}_{\lambda', \mu'}(q, 0). \) Since, \(\tilde{h}_\mu(q, 0) \tilde{h}_\mu'(q, 0) = (-q)^n q^{2n(\mu')} \prod_{s \in \mu, l(s) = 0} (1 -
Let $1 \leq k \leq n$, and let $\mu \in Par(n, r)$. Then,

$$T_{k,r} = \tilde{K}_{(n-k+1,1^{k-1})}\mu'(q).$$

**Proof.** Recall that

$$s_{k+1,1^{n-k-1}}[X \frac{1}{1-q}] = \sum_{r=k+1}^{n} T_{k+1,r} E_{n,r}(X) \frac{E_{n,r}(q)}{E_{n,r}(q)},$$

where

$$T_{k+1,r} = (-1)^{k} \sum_{i=0}^{k} (-1)^{i} q^{i \binom{r}{i}}.$$

Therefore, by Corollary 4.3 and Proposition 4.1 we have

$$\sum_{\mu} \frac{\tilde{K}_{(n-k+1,1^{k-1})}\mu'(q) \tilde{H}_{\mu'}(X; q)}{\tilde{h}_{\mu}(q, 0) \tilde{h}'_{\mu}(q, 0)} = s_{k,1^{n-k}}[X \frac{1}{1-q}]$$

$$= \sum_{r=k}^{n} T_{k,r} \sum_{\mu \in Par(n, r)} \frac{\tilde{H}_{\mu'}(X; q)}{\tilde{h}_{\mu}(q, 0) \tilde{h}'_{\mu}(q, 0)}$$

$$= \sum_{\mu \in \cup_{r=k}^{n} Par(n, r)} \frac{T_{k,r} \tilde{H}_{\mu'}(X; q)}{\tilde{h}_{\mu}(q, 0) \tilde{h}'_{\mu}(q, 0)}.$$

The theorem follows from comparison of the coefficients of $\tilde{H}_{\mu'}(X; q)$. □

**REFERENCES**


