Lexicographic Shellability of Partial Involutions

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Abstract

In this manuscript we study inclusion posets of Borel orbit closures on (symmetric) matrices. In particular, we show that the Bruhat poset of partial involutions is a lexicographically shellable poset. We determine which subintervals of the Bruhat posets are Eulerian, and moreover, by studying certain embeddings of the symmetric groups and their involutions into rook matrices and partial involutions, respectively, we obtain new shelling orders on the corresponding order complexes.

1 Introduction.

A $d$-dimensional simplicial complex $\Delta$ is called pure, if its maximal simplices all have dimension $d$. $\Delta$ is called shellable, if there exists a linear ordering $F_1, \ldots, F_k$ of the facets of $\Delta$ such that, for $2 \leq j \leq k$, the complex $\left( \bigcup_{i=1}^{j-1} F_i \right) \cap F_j$ is a pure subcomplex of $\Delta$ of dimension $\dim F_j - 1$.

Although its definition is not illuminating, the notion of shellability has remarkable topological consequences. For example, when shellable, the simplicial complex $\Delta$ has the homotopy type of a wedge of spheres. See [5]. The order complex, $\Delta(P)$ of a poset $P$ is the simplicial complex whose faces are the chains of $P$. One of our purposes in this manuscript is to prove the shellability of a certain order complex arising from an action of the invertible upper triangular matrices on the space of symmetric matrices.

Let $K$ denote an algebraically closed field of characteristic zero, let $M$ denote the affine variety of $n \times n$ matrices over $K$, and let $Y \subseteq M$ be a subvariety on which a group $G$ acts. Let $B(Y;G)$ denote the set of Zariski-closures of the $G$-orbits in $M$. We focus on the following examples:
• $Y = Q$, the space of symmetric matrices in $M$, and $G = B_n$, the Borel group of invertible upper triangular matrices acting on $Y$ via

$$x \cdot A = (x^{-1})^\top Ax^{-1},$$  \hspace{1cm} (1)

where $x^\top$ denotes the transpose of the matrix $x \in B_n$ and $A \in Q$.

• $Y = M$ and $G = B_n \times B_n$ acting on $Y$ via

$$(x, y) \cdot A = xAy^{-1},$$  \hspace{1cm} (2)

where $x, y \in B_n$ and $A \in M$.

**Definition 1.** The rook monoid $R_n$ is the finite monoid of 0,1-matrices with at most one 1 in each row and each column.

It was shown by Renner in [17] that $R_n$ parametrizes the orbits of the action (2) of $B_n \times B_n$ on $M$. The elements of $R_n$ are known as rook matrices. For action (1), Szechtman showed in [19] that each orbit closure in $B(Q; B_n)$ has a unique corresponding symmetric rook matrix in $R_n$.

**Definition 2.** A rook matrix is a partial involution if it satisfies a quadratic equation of the form

$$x^2 = e,$$  \hspace{1cm} (3)

where $e \in R_n$ is a diagonal matrix.

Note that $e$ depends on the particular involution $x$, and it is not the same for all involutions. For example, if $x$ is invertible, then $e$ is the identity matrix, and hence $x$ is an ordinary involution of the symmetric group $S_n$. For an arbitrary $e$, if $x$ satisfies (3), then $x$ is symmetric; moreover, $e$ determines the indices of the nonzero rows (hence, of the nonzero columns) of $x$.

The *Bruhat-Chevalley-Renner ordering* on rook matrices is defined by

$$r \leq t \iff B.nrB_n \subseteq B.ntB_n, \quad r, t \in R_n.$$  \hspace{1cm}

Here, the bar in our notation stands for the closure in the Zariski topology on $M$. The corresponding partial order on $P_n$, denoted by $\preceq$, is defined similarly; if $A$ and $A'$ are two $B_n$-orbit closures in $B(Q; B_n)$, and $r$ and $r'$ are two partial involutions representing $A$ and $A'$, respectively, then $r \preceq r' \iff A \subseteq A'$. 

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There is a simple combinatorial description for $\preceq$, which is due to Bagno and Cherniavsky [1]. Let $X = (x_{ij})$ be an $n \times m$ matrix. For $1 \leq k \leq n$ and $1 \leq l \leq m$, denote by $X_{kl}$ the upper-left $k \times l$ submatrix of $X$. The rank-control matrix of $X$ is the $n \times m$ matrix $R(X) = (r_{kl})$ with entries given by
$$r_{kl} = \text{rank}(X_{kl}),$$
for $1 \leq k \leq n$ and $1 \leq l \leq m$. For example, the rank-control matrix of the partial involution $x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is
$$R(x) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$  (4)

For two matrices $A = (a_{kl})$ and $B = (b_{kl})$ of the same size with integer entries, we write $A \preceq B$ if $a_{kl} \leq b_{kl}$ for all $k$ and $l$. It follows that $r \preceq r'$ if and only if $R(r) \preceq R(r')$.

Although $\preceq$ is more natural from a geometric point of view, we prefer to work with its dual, namely, the ordering $\leq$ obtained from $\preceq$ by reversing it: $r' \leq r \iff r \preceq r'$ for all $r, r' \in P_n$. The Hasse diagram of the dual order on partial involutions for $n = 3$ is depicted in Figure 4.

Our first main result is that

**Theorem 3.** The order complex of the poset $(P_n, \leq)$ of partial involutions is shellable.

Let $S_n$ denote the symmetric group of permutations, which is contained in $R_n$ as the group of invertible rook matrices. In increasing order of generality, the articles [11], [15], and [4] show that $(S_n, \leq)$ is a lexicographically shellable poset. Generalizing this result to $R_n$, [6] showed that $R_n$ is a lexicographically shellable poset.

A graded poset $(P, \leq)$ with rank function $\rho : P \rightarrow \mathbb{N}$ is called Eulerian, if for all $x \leq y$ the equality
$$|\{z \in [x, y] : \rho(z) \text{ is odd}\}| = |\{z \in [x, y] : \rho(z) \text{ is even}\}|$$
holds. Let $I_n \subset P_n$ denote the subset consisting of invertible involutions. It is shown by Incitti in [13] that $I_n$ with its “opposite inclusion ordering” is not only lexicographically
shellable but also Eulerian. Unfortunately, neither $R_n$ nor $P_n$ is Eulerian, so, we direct our attention to certain important subposets of them.

Let $H$ denote the group of invertible elements of a monoid $N$. It is important for semigroup theorists to understand the structure of orbits of $H$ on $N$ for various actions. In this regard, we consider the following action of $S_n \times S_n$ on $R_n$:

$$(x, y) \cdot z = xzy^{-1}, \text{ for all } z \in R_n, \ x, y \in S_n.$$  \hspace{1cm} (5)

There is a natural restriction of this action to its diagonal subgroup $S_n \simeq \Delta S_n \hookrightarrow S_n \times S_n$ on partial involutions:

$$y \cdot t = (y^{-1})^\top ty^{-1}, \text{ for } t \in P_n, \ y \in S_n.$$  \hspace{1cm} (6)

Let $R_{n,k} \subset R_n$ denote the set of rook matrices with $k$ nonzero entries, and let $P_{n,k} = R_{n,k} \cap P_n$. Any orbit of (5) is equal to one of $R_{n,k}$ for some $k$, and similarly any orbit of (6) is equal to one of $P_{n,k}$ for some $k$.

The unions $\bigcup_{i \leq k} R_{n,i}$ and $\bigcup_{i \leq k} P_{n,i}$ parametrize Borel orbits in certain determinantal varieties. Indeed, a determinantal variety associated with the vector space $\C^n$ is, by definition, the affine algebraic variety $X$ consisting of all linear maps of rank at most $k$, for some $k \in \N$. See Section 5.2 in [14]. Let $C_k$ denote $\bigcup_{i \leq k} R_{n,i}$ and observe that $\bigcup_{r \in C_k} B_n r B_n$ consists of all matrices of rank at most $k$, hence it is isomorphic to $X$. Although they are significantly different from each other, $R_{n,k}$ and $P_{n,k}$ share many important properties. For example, both of them have the same smallest and largest elements. In fact, much more is true.

Given a positive integer $n$, let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

**Theorem 4.** For $n \geq 1$ and $k \in [n]$, the subposets $R_{n,k} \subseteq R_n$ and $P_{n,k} \subseteq P_n$ are Eulerian if and only if $k = n$ or $k = n - 1$.

The proof of Theorem 4 relies on the following intriguing result:

**Theorem 5.** For $n \geq 1$,

1. $(R_{n,n-1} \cup R_{n,n}, \leq)$ is isomorphic to the poset $(S_{n+1}, \leq)$, and
2. $(P_{n,n-1} \cup P_{n,n}, \leq)$ is isomorphic to the poset $(I_{n+1}, \leq)$.

As a corollary of Theorems 3 and 5 together with the main result of [6], we obtain new shelling orders on $S_{n+1}$ and $I_{n+1}$ induced from their imbedding into $R_n$ and $P_n$, respectively.
We conclude our introduction by giving some references to recent developments. It is well known that the $B_n$-orbits under congruence action on invertible, $2n \times 2n$ skew-symmetric matrices are parametrized by fixed-point-free involutions of $S_{2n}$. In [10], extending this fact to the space of all skew-symmetric matrices, Cherniavsky showed that the Borel orbits are parametrized by the partial fixed-point-free involutions in $R_{2n}$. Furthermore, in the same paper Cherniavsky gave a combinatorial description of the inclusion ordering of Borel orbit closures in terms of rank control matrices of partial fixed-point-free involutions. In [7], the authors together with Cherniavsky showed that the inclusion poset of invertible fixed-point-free involutions has a shellable order complex. Recently, in [8], the same authors showed that the order complex of the inclusion poset of partial fixed-point-free involutions is shellable, also.

The organization of our paper is as follows. In Section 2 we introduce our notation and provide preliminaries. In Section 3 we study covering relations of the opposite order $\leq$. In Section 4 we prove Theorem 3, and finally, in Section 5 we prove Theorems 4 and 5.

2 Background.

2.1 Lexicographic shellability.

Recall that the set of all finite chains in a poset forms a simplicial complex, called the order complex of the poset. Let $\Delta(Y)$ denote the order complex of $(B(Y; G), \subseteq)$. It is known that the inclusion poset of Borel orbits in a variety have a pure order complex (see Section 8.9, Exercise 12 in [18]). Here, “pureness” is equivalent to all maximal chains of $B(Y)$ having the same length. Our first main result is that $\Delta(Q)$ is a shellable complex. In fact, we prove a much stronger statement; the poset $(B(Q; B_n), \subseteq)$ is “lexicographically shellable.” Introduced by Björner in [2] and advanced by Björner and Wachs in [4], the notion of lexicographic shellability amounts to finding a suitable labeling of the edges of the Hasse diagram of the poset under consideration. Thus, associated with each saturated chain is a sequence of labels, which provides an ordering of the facets of the simplicial complex $\Delta(Y)$. Let us mention in passing that A. Hultman, in [12], without proving lexicographic shellability, obtains all of its topological consequences for the BC-order on ‘twisted involutions’ of Coxeter groups.

In the literature there are various versions of lexicographic shellability. The one we introduce here, namely ‘EL-shellability’, is known to imply all of the others. As an example of a weaker lexicographic shellability notion, we have the notion of “CL-shellability,” which is introduced in [4]. In the same article, Björner and Wachs show
that BC-order on all Coxeter groups, as well as on all sets of minimal-length coset representatives (quotients) in Coxeter groups are “dual CL-shellable.” In another setting, for semigroups, Putcha showed that “J-classes in Renner monoids” are CL-shellable. See [16].

We continue with the definition of EL-shellability. Let $P$ be a finite poset with a maximum and a minimum element, denoted by $\hat{1}$ and $\hat{0}$, respectively. We assume that $P$ is graded of rank $n$. In other words, all maximal chains of $P$ have equal length $n$. Denote by $C(P)$ the set of covering relations

$$C(P) = \{(x, y) \in P \times P : y \text{ covers } x\}.$$ 

An edge-labeling on $P$ is a map $f = f_{P, \Gamma} : C(P) \to \Gamma$ into some totally ordered set $\Gamma$. The Jordan-Hölder sequence (with respect to $f$) of a maximal chain $c$ with elements $x_0, \ldots, x_n$ in increasing order is the $n$-tuple $f(c)$ defined by

$$f(c) = (f((x_0, x_1)), f((x_1, x_2)), \ldots, f((x_{n-1}, x_n))) \in \Gamma^n.$$ 

Fix an edge labeling $f$, and such a maximal chain $c$. We call both $c$ and its image $f(c)$ increasing if

$$f((x_0, x_1)) \leq f((x_1, x_2)) \leq \cdots \leq f((x_{n-1}, x_n))$$

holds in $\Gamma$.

Let $k > 0$ be a positive integer and let $\Gamma^k$ denote the $k$-fold cartesian product $\Gamma^k = \Gamma \times \cdots \times \Gamma$, totally ordered with respect to the lexicographic ordering. An edge labeling $f : C(P) \to \Gamma$ is an EL-labeling if

1. in every interval $[x, y] \subseteq P$ of rank $k > 0$ there exists a unique maximal chain $c$ such that $f(c) \in \Gamma^k$ is increasing, and

2. the Jordan-Hölder sequence $f(c) \in \Gamma^k$ of the unique chain $c$ from (1) is the smallest among the Jordan-Hölder sequences of maximal chains $x = x_0 < x_1 < \cdots < x_k = y$.

A poset $P$ is called EL-shellable, if it has an EL-labeling.

2.2 **Rook matrices and their enumeration.**

We set up our notation for rook matrices and establish a preliminary enumerative result.
Let \( x = (x_{ij}) \in R_n \) be a rook matrix of size \( n \). Define the sequence \((a_1, \ldots, a_n)\) by

\[
a_j = \begin{cases} 
0 & \text{if the } j\text{th column consists of zeros,} \\
i & \text{if } x_{ij} = 1.
\end{cases}
\]  

(7)

Abusing notation, we denote both the matrix and the sequence \((a_1, \ldots, a_n)\) by \( x \). For example, the associated sequence of the partial permutation matrix

\[
x = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

is \( x = (3, 0, 4, 0) \).

Once \( n \) is fixed, a rook matrix \( x \in R_n \) with \( k \) nonzero entries is called a \( k \)-rook.

Observe that the number of \( k \)-rooks is given by the formula

\[
|R_{n,k}| = k! \cdot \binom{n}{k}^2.
\]  

(8)

Indeed, to determine a \( k \)-rook, we first choose \( n - k \) all-0 zero rows and \( n - k \) all-0 columns. This is done in \( \binom{n}{n-k}^2 \) ways. Next we specify the nonzero entries of the \( k \)-rook. Since deleting the zero rows and columns results in a permutation matrix of size \( k \), there are \( k! \) possibilities. Hence, the formula follows.

Let \( \tau_n \) denote the number of invertible partial involutions. By default, we set \( \tau_0 = 1 \).

There is no closed formula for \( \tau_n \), however, there is a simple recurrence that it satisfies;

\[
\tau_{n+1} = \tau_n + (n-1)\tau_{n-1} \quad (n \geq 1).
\]  

(9)

To see (9), we argue as follows: For a given \( x \in I_{n+1} \), let \( x_r, x_l, x_r' \) and \( x_l' \) denote, respectively, the last row, the last column, the row of the nonzero entry in the last column, and the column of the nonzero entry in the last row of \( x \). Note that, \( x_r = x_r' \) (hence, \( x_l = x_l' \)) if and only if the \((n+1),(n+1)\)th entry of \( x \) is nonzero. With this in mind, it is easy to verify that the cardinality of the multiset of involutions obtained by deleting \( x_r, x_r', x_l, x_l' \) from \( x \) is equal to the right hand side of (9).

There is a similar recurrence satisfied by the number of invertible \( n \)-rooks (permutations);

\[
(n+1)! = n! + n^2 \cdot (n-1)! \quad (n \geq 1).
\]  

(10)

It follows that
Lemma 6. For all $n \geq 1$,

1. $|R_{n,n-1} \cup R_{n,n}| = (n + 1)!$.
2. $|P_{n,n-1} \cup P_{n,n}| = \tau_{n+1}$.

Proof. The first assertion follows from equations (10) and (8). The second assertion follows from equation (9) and the fact that $|P_{n,n-1}| = (n - 1)\tau_{n-1}$. □

2.3 An $EL$-labeling of invertible involutions.

In [13], Incitti showed that the poset of invertible involutions is $EL$-shellable. Let us briefly recall his arguments.

For a permutation $\sigma \in S_n$, a rise of $\sigma$ is a pair $(i,j) \in [n] \times [n]$ such that $i < j$ and $\sigma(i) < \sigma(j)$.

A rise $(i,j)$ is called free if there is no $k \in [n]$ such that $i < k < j$ and $\sigma(i) < \sigma(k) < \sigma(j)$.

For $\sigma \in S_n$, define its fixed point set, its exceedance set and its defect set to be

\[
I_f(\sigma) = \text{Fix(\sigma)} = \{i \in [n] : \sigma(i) = i\}, \\
I_e(\sigma) = \text{Exc(\sigma)} = \{i \in [n] : \sigma(i) > i\}, \\
I_d(\sigma) = \text{Def(\sigma)} = \{i \in [n] : \sigma(i) < i\},
\]

respectively.

The type of a rise $(i,j)$ is defined to be the pair $(a,b)$ if $i \in I_a(\sigma)$ and $j \in I_b(\sigma)$, for some $a,b \in \{f,e,d\}$. We call a rise of type $(a,b)$ an $ab$-rise. Two kinds of $ee$-rises have to be distinguished from each other; an $ee$-rise is called crossing if $i < \sigma(i) < j < \sigma(j)$, and it is called non-crossing if $i < j < \sigma(i) < \sigma(j)$. The rise $(i,j)$ of an involution $\sigma \in I_n$ is called suitable, if it is free and if its type is one of the following: $(f,f), (f,e), (e,f), (e,e), (e,d)$. We depict these possibilities in the first two columns in Figure 1, below.

It is easy to check that the involution $\tau$ that is on the rightmost column in Figure 1 below covers $\sigma$. In this case, the covering relation is called a covering transformation of type $(i,j)$, and $\tau$ is denoted by $\text{ct}_{(i,j)}(\sigma)$. In [13], Incitti showed that covering transformations exhaust all possible covering relations in $I_n$. Moreover, he showed that the labeling that assigns $(i,j)$ to the pair $(\sigma,\text{ct}_{(i,j)}(\sigma))$ is an $EL$-labeling for $I_n$.

In the next section we extend Incitti's result to the set of all partial involutions.
Figure 1: Covering transformations $\sigma \leftarrow \tau = \text{ct}_{(i,j)}(\sigma)$ of $I_n$. 
3 Covering relations of partial involutions.

Covering relations in \( P_n \) depend on a numerical invariant associated with the rank control matrices. For any non-negative integer \( k \), define \( r_{0,k} \) to be 0. For a rank-control matrix \( R(X) = (r_{ij}) \), define

\[
D(x) = \#\{(i,j) | 1 \leq i \leq j \leq n \text{ and } r_{ij} = r_{i-1,j-1}\}.
\]

For example, if \( R(x) \) is as in (4), then \( D(x) = \#\{(2,2), (2,3)\} = 2 \). In [1], Bagno and Cherniavsky prove that, in \( (P_n, \preceq) \),

\[
x \text{ covers } y \iff R(x) \preceq_R R(y) \text{ and } D(x) = D(y) + 1. \tag{11}
\]

However, we need a finer classification of the covering types. The notion of suitable rise on involutions extends to the partial permutations \( (P_n, \preceq) \), verbatim. Of course, there are additional covering relations. In this section we exhibit all of them. Before we present our result, we illustrate it by examples.

When two partial involutions \( x \) and \( y \) have the same zero rows and zero columns, the covering relation \( x \to y \) is not different than the invertible case. For example,

**Example 7.** By (11), we easily verify that

\[
y = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\text{ is covered by } x = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}.
\]

Notice that \( x \to y \) if and only if the invertible involution \( \tilde{x} \), that is obtained from \( x \) by removing the rows and columns of \( x \) with no nonzero entries, covers the invertible involution \( \tilde{y} \) that is obtained from \( y \) by removing its rows and columns with zeros only.

Moving down a nonzero entry along the diagonal gives a covering relation:

**Example 8.**

\[
y = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\text{ is covered by } x = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Similarly,

\[
y = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\text{ is covered by } x = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
Another type of covering relation is obtained by the moving of off-diagonal pairs 
\((i, j)\) and \((j, i)\), where \(i > j\) to down/right, or to right/down available positions.

**Example 9.** There are two cases:

1. \(y = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\) is covered by 
   \(x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\),

2. \(y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\) is covered by 
   \(x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\).

When a down/right move is performed on \(y\) (as in part 2. of Example 9), there 
may not be any available positions to place the nonzero entries of \(x\). In this case, the 
pushed entries are placed on the diagonal. If there are no available diagonal entries for 
both of the 1’s, then one of them is pushed out of the matrix.

**Example 10.** Once again, there are two moves of similar nature:

1. \(y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\) is covered by 
   \(x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\).

2. \(y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) is covered by 
   \(x = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\).

In the light of the above examples, we label a covering relation \(x \rightarrow y\) in \(P_n\) as 
follows.

**Definition 11.** 1. As in Example 7, if the covering relation \(x \rightarrow y\) is derived from 
the covering relation \(\tilde{x} \rightarrow \tilde{y}\) of invertible involutions that are obtained from \(x\) and 
\(y\), respectively, then we use the labeling \(\tilde{x} \rightarrow \tilde{y}\) as defined in [13].

2. If the covering relation results from a move as in Example 8, namely from a 
diagonal push where the element that is pushed from is at the position \((i, i)\), then 
we label it by \((i, i)\).

3. Suppose \(x \rightarrow y\) is as in Example 9, or 10. Observe that, in all of these covering 
relations, one of the 1’s is pushed down and the other is pushed right. Let \(i\) denote 
the column index of the first 1 that is pushed to the right, and let \(j\) denote the 
index of the resulting column. We label such a covering by \((i, j)\).
To illustrate the third labeling let us present a few more examples.

Example 12.

\[
y = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\quad \text{is covered by } x = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{pmatrix}
\]

The corresponding labeling here is (3, 5).

Example 13.

\[
y = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\quad \text{is covered by } x = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
\]

The corresponding labeling here is (1, 3).

Example 14.

\[
y = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\quad \text{is covered by } x = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

The corresponding labeling here is (2, 3).

Definition 15. If \(x\) covers \(y\) with label \((i, j)\), then we refer to it as an \((i, j)\)-covering and say that \(y\) is obtained from \(x\) by an \((i, j)\)-move. More briefly, we call a covering relation a \(c\)-cover, if it is derived from an involution; a \(d\)-cover if it is obtained by a shift of a diagonal element; an \(r\)-cover if it is derived from a right/down, or from a down/right move. The corresponding moves of 1’s are referred to as \(c\)-, \(d\)- and \(r\)-moves.

In Figure 2, we illustrate the labelings of the covering relations for \(P_3\) whose elements are depicted in one-line notation.
Figure 2: The $EL$-labeling of $P_3$. 
Lemma 16. Let $x$ and $y$ be two partial involutions. $x$ covers $y$ if and only if one of the following is true:

1. $x$ is obtained from $y$ by a $c$-move as in Example 7.

2. Without removing a suitable rise, $x$ is obtained from $y$ by one of the following moves:

   (a) a $d$-move, as in Example 8,
   (b) an $r$-move, as in Example 9, or as in Example 10.

Before we start the proof, let us illustrate by an example, what it means to remove a suitable rise:

Example 17. Let $y = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and let $x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$. The partial involution $x$ is obtained from $y$ by an $r$-move, however, it removes the suitable rise $(1,3)$. Therefore, it is not a covering relation.

Proof. Comparing the rank-control matrices $R(\cdot)$ as well as the invariants $D(\cdot)$ of $x$ and $y$, the “if” direction of the claim is straightforward to verify.

We prove the “only if” direction by contraposition. To this end let $x$ denote a partial involution that covers $y \in P_n$, and $x$ is not obtained by one the moves as in 1., 2.(a), or 2.(b).

Since 1. does not hold, $x$ has a row of zeros with the smallest index $i \in [n]$ such that the $i$th row of $y$ contains a nonzero entry. Notice that, if there is a zero row for both $x$ and $y$ with the same index, then removing this row and the corresponding column does not have any effect on the remaining entries of the rank-control matrices. Therefore, we assume that neither $x$ nor $y$ has a zero row before the $i$th row.

There are two subcases; 1) the nonzero entry of $y$ on its $i$th row occurs before the $i$th column, 2) it occurs on, or after the $i$th column. The idea of the proof of the second case is similar to that of the first, so we skip it. We proceed with the first case.

Let $1 \leq k < i$ denote the index of the row of $y$ with a 1 on its $i$th entry. Let $\Gamma$ denote the set of positions $(r, s)$ of nonzero entries of $x$ satisfying $k \leq r \leq n$ and $i < s \leq n$. Observe that $\Gamma$ cannot be empty (the upper $k \times n$ portions of both of $y$ and $x$ are of rank $k$). Let $(r, s)$ be the element of $\Gamma$ with the smallest second coordinate. Unless $r = s$, we define $x_1$ to be the matrix obtained from $x$ by moving the nonzero entries at the positions $(r, s)$ and $(s, r)$ to the positions $(r, i)$ and $(i, r)$. If $r = s$, then $x_1$ is
defined by moving the nonzero entry to the \((i, i)\)th position. We claim that \(y < x_1 < x\). Indeed, since \(x_1\) is obtained from \(x\) by reverse of the one of the moves 2.(a) or 2.(b), the second inequality is clear. The first inequality follows immediately from checking the corresponding rank-control matrices of \(x, x_1\) and of \(y\). Since \(x_1\) is different \(y\), we conclude that \(x\) does not cover \(y\), hence the proof is complete.

We illustrate the proof of 1) in Example 18, below.

Example 18. Let

\[
y = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\text{ and } x = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

Clearly, \(i = 5, k = 1\). The rank-control matrices of \(y\) and \(x\) are given by

\[
R(y) = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\
0 & 1 & 2 & 2 & 3 & 3 & 3 & 3 \\
0 & 1 & 2 & 2 & 3 & 3 & 3 & 4 \\
1 & 2 & 3 & 3 & 4 & 4 & 4 & 5 \\
1 & 2 & 3 & 3 & 4 & 5 & 5 & 6 \\
1 & 2 & 3 & 3 & 4 & 5 & 5 & 6 \\
1 & 2 & 3 & 4 & 5 & 6 & 6 & 7 \\
\end{pmatrix}
\text{ and } R(x) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\
0 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\
0 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\
0 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\
1 & 2 & 3 & 3 & 3 & 4 & 4 & 5 \\
1 & 2 & 3 & 3 & 3 & 4 & 5 & 6 \\
1 & 2 & 3 & 4 & 4 & 5 & 6 & 7 \\
\end{pmatrix}.
\]

In this case,

\[
x_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\text{ and } R(x_1) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\
0 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\
0 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\
0 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\
1 & 2 & 3 & 3 & 3 & 4 & 4 & 5 \\
1 & 2 & 3 & 3 & 3 & 4 & 5 & 6 \\
1 & 2 & 3 & 4 & 4 & 5 & 6 & 7 \\
\end{pmatrix}.
\]
4 EL-labeling of $P_n$.

In this section we prove that the edge labeling introduced in the previous section is an EL-labeling.

**Proposition 19.** Let $F : C(P_n) \rightarrow \Gamma$ denote the labeling defined as in Definition 15. Let $[x, y]$ be an interval in $P_n$. If $c$ is a maximal chain with entries $x = x_0, \ldots, x_k = y$ in increasing order such that the Jordan-Hölder sequence $F(c)$ is lexicographically smallest among all Jordan-Hölder sequences (of maximal chains of $[x, y]$) in $\Gamma^k$, where $\Gamma$ is the lexicographic ordering on $[n] \times [n]$, then

$$F((x_0, x_1)) \leq F((x_1, x_2)) \leq \cdots \leq F((x_{k-1}, x_k)).$$

**Proof.** Assume that (12) is not true, so there exist three consecutive terms

$$x_{t-1} < x_t < x_{t+1}$$

in $c$ such that $F((x_{t-1}, x_t)) > F((x_t, x_{t+1}))$. We have nine cases to consider.

<table>
<thead>
<tr>
<th>Case</th>
<th>Type of $x_{t-1}, x_t$</th>
<th>Type of $x_t, x_{t+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$c$</td>
<td>$c$</td>
</tr>
<tr>
<td>2</td>
<td>$d$</td>
<td>$d$</td>
</tr>
<tr>
<td>3</td>
<td>$d$</td>
<td>$c$</td>
</tr>
<tr>
<td>4</td>
<td>$c$</td>
<td>$d$</td>
</tr>
<tr>
<td>5</td>
<td>$r$</td>
<td>$r$</td>
</tr>
<tr>
<td>6</td>
<td>$d$</td>
<td>$r$</td>
</tr>
<tr>
<td>7</td>
<td>$r$</td>
<td>$d$</td>
</tr>
<tr>
<td>8</td>
<td>$r$</td>
<td>$c$</td>
</tr>
<tr>
<td>9</td>
<td>$c$</td>
<td>$r$</td>
</tr>
</tbody>
</table>

In each of these nine cases, we either produce an immediate contradiction by showing that we can interchange the two moves, or we construct an element $z \in [x, y]$ which covers $x_{t-1}$, and such that $F((x_{t-1}, z)) < F((x_{t-1}, x_t))$. Since we assume that $F(c)$ is the lexicographically first Jordan-Hölder sequence, the existence of $z$ is a contradiction, too.

**Case 1:** Straightforward from the fact that type $c$ covering relations have identical labelings with Incitti’s [13].

**Case 2:** Suppose that the first move is labeled $(i, i)$ and the second one $(j, j)$ with $j < i$. If the two moves are not interchangeable then $(j, i)$ is a legal $c$-move in $x_{t-1}$. Since $(j, i)$ is lexicographically smaller than $(i, i)$, we derive a contradiction.

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Case 3: Let \((i, i)\) be moved to \((j, j)\) in the first step (type \textit{d} move), hence \(i < j\). If the following \textit{c}-move does not involve the entry at \((j, j)\), then either the \textit{c}- and the \textit{d}-move commute with each other, or the rise for the \textit{c}-move is not free in \(x_{t-1}\). In that case there has to be an \textit{ef}-rise involving the entry at the position \((i, i)\). This \textit{ef}-rise has a smaller label than \((i, i)\), which is a contradiction. Thus we may assume that the \textit{c}-move involves the entry at the \((j, j)\)th position. Then the \textit{c}-move has to come from either an \textit{ff}-, an \textit{fe}-, or an \textit{ef}-rise.

\textit{ff} is not possible: Let \((a, b)\) the corresponding label. The move involves the entry at the \((j, j)\)th position if either \(a = j\) or \(b = j\). If \(a = j\) then \((a, b) > (i, i)\) and the labels are increasing. If \(b = j\), then we must have \(a < i\) for \((a, b) < (i, i)\). Therefore, there is a legal \textit{c}-move \((a, i)\) in \(x_{t-1}\) has a smaller label than \((i, i)\).

\textit{fe} is not possible since \((j, b)\) is greater than \((i, i)\).

Finally, \textit{ef} is not possible: Let \((k, j)\) be the label of the \textit{c}-move. If \((k, i)\) is a suitable rise in \(x_{t-1}\), then \((k, i) < (i, i)\). If \((k, i)\) is not a suitable rise in \(x_{t-1}\), let \((j, j), (k, l)\), and \((l, k)\) denote the entries involved in the \textit{c}-move where \(l < k\). Then \(l < i < k\) and \((l, j) < (i, i)\). \((l, j)\) is a legal \textit{r}-move in \(x_{t-1}\) with a smaller label than \((i, i)\). This concludes case 3.

Case 4: This is not possible since no \textit{c}-move places a 1 on the diagonal such that moving this 1 gives rise to a smaller labeling than the \textit{c}-move. Note that if there is a 1 on the diagonal before the \textit{c}-move takes place, then moving this 1 first creates an element \(z\) with covering label lexicographically smaller than that of the \textit{c}-move. Thus we are done with this case.

Case 5: Let the first move be labeled \((i, j)\) and the second \((k, l)\). It is clear that if \(k = j\), then we have an increasing sequence. If \(k = i\), then we can switch the order of the moves. If \(k \neq \{i, j\}\), then, if possible, we perform the second move first. If it is not possible to interchange the order of the \textit{r}-moves, then by the first \textit{r}-move a suitable rise is removed. But then the corresponding \textit{c}-cover has a smaller label in \(x_{t-1}\) than the \textit{r}-move.

Case 6: We either perform the \textit{r}-move first if possible, or perform the \textit{c}-cover corresponding to the suitable rise removed by the \textit{d}-move which has a smaller label than the \textit{d}-move in \(x_{t-1}\).

Case 7: Similar to Case 6 so we omit the proof.

Case 8: The \textit{c}-move has to include the elements moved by the previous \textit{r}-move since otherwise the \textit{c}-move can be performed first.

If the suitable rise is created by the \textit{r}-move then the label of the \textit{r}-move is smaller than the label of the \textit{c}-move. Otherwise, there is a suitable rise in \(x_{t-1}\) involving the elements moved by the \textit{r}-move. But the \textit{c}-move corresponding to this suitable rise has
Case 9: If the $r$-move does not involve an element moved by the $c$-move then perform the $r$-move first. If this is not possible then a suitable rise is removed by moving it. The $c$-move corresponding to this suitable rise has a smaller label than the other $c$-move.

If the $r$-move involves an element that is placed at this position by the preceding $c$-move, then we proceed to exhibit every $c$-move to exclude all of them:

**ff**: The label of $c$-move is $(i, j)$. The smaller $r$-move involving a new element can only be $(i, k)$ with $k < j$. But then $(i, i)$ is possible in $x_{l-1}$ and $(i, i) < (i, j)$.

**fe**: Similar to **ff** so we omit the proof.

**ef**: The label of $c$-move is $(i, j)$. The smaller $r$-move involving a new element can only be $(i, k)$ with $k < j$. Then $(i, i)$ is possible in $x_{l-1}$ and $(i, k) < (i, j)$.

Non-crossing $ee$, crossing $ee$ and $ed$ are similar to $ef$ so we omit the proof.

**Proposition 20.** For any interval $[x, y] \subseteq P_n$, there exists a unique maximal chain $x = x_0 < \cdots < x_k = y$ with $F((x_0, x_1)) \leq \cdots \leq F((x_{k-1}, x_k))$.

**Proof.** We already know that the lexicographically first chain is increasing. Therefore, it is enough to show that there is no other increasing chain. We prove this by induction on the length of the interval $[x, y]$. Clearly, if $y$ covers $x$, there is nothing to prove. So, we assume that for any interval of length $k$ there exists a unique increasing maximal chain.

Let $[x, y] \subseteq P_n$ be an interval of length $k + 1$, and let $c$ be the maximal chain with elements $x = x_0, x_1, \ldots, x_k, x_{k+1} = y$ in increasing order such that $F(c)$ is the lexicographically first Jordan-Hölder sequence in $\Gamma^{k+1}$. Assume that there exists another increasing chain $c'$ with elements $x = x_0, x'_1, \ldots, x'_k, x_{k+1} = y$ in increasing order. Since the length of the chain $x'_1 < \cdots < x'_k < x_{k+1} = y$ is $k$, by the induction hypotheses, it is the lexicographically first chain between $x'_1$ and $y$. We are going to find contradictions to each of the following possibilities:

- **Case 1**: type($x_0, x_1$) = $c$, and type($x_0, x'_1$) = $c$,
- **Case 2**: type($x_0, x_1$) = $d$, and type($x_0, x'_1$) = $d$,
- **Case 3**: type($x_0, x_1$) = $d$, and type($x_0, x'_1$) = $c$,
- **Case 4**: type($x_0, x_1$) = $c$, and type($x_0, x'_1$) = $d$,
- **Case 5**: type($x_0, x_1$) = $r$, and type($x_0, x'_1$) = $r$,
- **Case 6**: type($x_0, x_1$) = $d$, and type($x_0, x'_1$) = $r$,
- **Case 7**: type($x_0, x_1$) = $r$, and type($x_0, x'_1$) = $d$,
- **Case 8**: type($x_0, x_1$) = $r$, and type($x_0, x'_1$) = $c$,
Case 9: type($x_0, x_1$) = $c$, and type($x_0, x'_1$) = $r$.

In each of these cases, we construct a partial involution $z$ covering $x'_1$ such that $F((x'_1, z)) < F((x'_1, x'_2))$. In all of the cases, by comparing the rank-control matrices, it is tedious but straightforward to check that $z$ belongs to the interval $[x, y]$. We verify this for Case 3, and skip the proofs for remaining cases since they are similar. Hence, we obtain a contradiction to the induction hypothesis.

Case 1: Proved in [13].

Case 2: $F(x_0, x_1) = (i, i) < F(x_0, x'_1) = (j, j)$ with $i < j$. In $x'_1$, $(i, i)$ is a legal covering move, so we apply to define $z$. The labeling of the (covering) pair $(x'_1, z)$ gives us the desired contradiction, if we show that $z \in [x, y]$.

Clearly, $x_0 < z$. To see that $z < y$, it is enough to compare their rank-control matrices. Since both $x_1$ and $x'_1$ cover $x_0$ with a $d$-move, the rank-control matrix of $x_0$ differs from that of $x_1$ only at the $i$th row and column, and it differs from that of $x'_1$ only at the $j$th row and column. The rest of the entries of the corresponding rank-control matrices are the same. Writing the corresponding rank-control matrix of $z$, we see that it is different than that of $x_0$ only at the $i$th and the $j$th rows and columns, hence $R(z) > R(y)$, hence $z < y$.

Case 3: $F(x_0, x_1) = (i, i) < F(x_0, x'_1) = (j, k)$. There are two cases to consider: $i = j$ and $i < j$. If $i < j$, then we reverse the order of the $d$ and $c$ moves and get to the same contradiction as in Case 2. If $i = j$, then $k \neq i + 1$ since otherwise $(i, i)$ is not a possible move for $x_0$. The $d$-cover moves $(i, i)$th entry to the $(l, l)$th, where $l < k$. But then $(i, l)$ is a legal move for $x'_1$ and $(i, l) < (i, k)$.

Case 4: $F(x_0, x_1) = (i, j) < F(x_0, x'_1) = (k, k)$. There are two cases to be considered: $j = k$ and $j \neq k$. If $j \neq k$ then $k \not\in [i, j]$ since otherwise $(i, j)$ is not a suitable rise. Hence $k > j$. But this means the two covering moves are interchangeable. We get to the same contradiction as in the previous cases. If $j = k$ then $(i, j)$ is a legal $r$-move of $x'_1$ with $(i, j) < (k, k)$. Contradiction.

Case 5: $F(x_0, x_1) = (i, j) < F(x_0, x'_1) = (k, l)$. $k > i$ since there is at most one legal $r$-move of each element. We also have $j < l$ since otherwise either $(i, k)$ or $(i, l)$ is a suitable rise with a label less than $(i, j)$. We have two cases to consider:

(a) $i < j < k < l$
(b) \( i < k < j < l \)

In case of (a), the two moves are interchangeable. In case of (b), \((i, k)\) is a suitable rise of \(x_0\) with \((i, k) < (i, j)\).

Case 6: \( F(x_0, x_1) = (i, i) < F(x_0, x_1') = (j, k) \). \( j \neq i \) by construction. Hence \( i < j < k \) and therefore \((j, k)\) does not effect the move \((i, i)\) and we have a contradiction as before.

Case 7: \( F(x_0, x_1) = (i, j) < F(x_0, x_1') = (k, k) \). \( k > j \) since otherwise a suitable rise is removed by \((i, j)\). But then \((i, j)\) is a legal move of \(x_1'\).

Case 8: \( F(x_0, x_1) = (i, j) < F(x_0, x_1') = (k, l) \). Two cases need to be considered: \( i < k \) and \( i = k \). \( i < j < l \). If \( i < k \) then \( j < k \) since otherwise a suitable rise would have been removed by \((i, j)\). But this means that \((i, j)\) is a legal move of \(x_1'\). If \( i = k \) then the \(c\)-move corresponds to an \(ef\), non-crossing \(ee\), crossing \(ee\) or a \(ed\) rise. In each of these cases \((i, j)\) is a legal move of \(x_1'\).

Case 9: \( F(x_0, x_1) = (i, j) < F(x_0, x_1') = (k, l) \). We have two cases to consider:

(a) \( i = k, i < j < l \)

(b) \( i < k \)

Note that (a) does not occur because then the \(r\)-move removes a suitable rise, hence, it is not a covering relation. We continue with (b). Since we have \( i < k < l \) and \( i < j \), we consider \( i < j < k < l, i < k < j < l \) and \( i < k < l < j \). In all these cases the \(c\)- and the \(r\)-moves are interchangeable, hence we obtain a contradiction. This ends the proofs of our claims.

Combining previous two propositions we obtain our first main result.

**Theorem 3.** The poset of partial involutions is lexicographically shellable.

**Example 21.** The dashed lines in Figure 2 gives the increasing maximal chain of the EL-labeling of \(P_3\).
5 Eulerian intervals.

In this section we prove Theorems 4 and 5.

There is a concrete way to compare two rooks given in one line notation in Bruhat-Chevalley-Renner ordering. For an integer valued vector \(a = (a_1, \ldots, a_n) \in \mathbb{Z}^n\), let \(\tilde{a} = (a_{\alpha_1}, \ldots, a_{\alpha_n})\) be the rearrangement of the entries \(a_1, \ldots, a_n\) of \(a\) in a non-increasing fashion:

\[
a_{\alpha_1} \geq a_{\alpha_2} \geq \cdots \geq a_{\alpha_n}.
\]

The containment ordering, \(\leq_c\), on \(\mathbb{Z}^n\) is then defined by

\[
a = (a_1, \ldots, a_n) \leq_c b = (b_1, \ldots, b_n) \iff a_{\alpha_j} \leq b_{\alpha_j} \text{ for all } j = 1, \ldots, n.
\]

Example 22. Let \(x = (4, 0, 2, 3, 1)\), and let \(y = (4, 3, 0, 5, 1)\). In this case, \(x \leq_c y\), because

\[
\tilde{x} = (4, 3, 2, 1, 0) \quad \text{and} \quad \tilde{y} = (5, 4, 3, 1, 0).
\]

For \(k \in [n]\), the \(k\)th truncation \(a(k)\) of \(a = (a_1, \ldots, a_n)\) is defined to be

\[
a(k) = (a_1, a_2, \ldots, a_k).
\]

Let \(v = (v_1, \ldots, v_n)\) and \(w = (w_1, \ldots, w_n)\) be two rooks in \(R_n\). It is shown in [9] that

\[
v \leq w \iff \tilde{v(k)} \leq_c \tilde{w(k)} \text{ for all } k = 1, \ldots, n.
\]

Example 23. If \(x = (0, 1, 2, 3, 4)\) and \(y = (4, 3, 2, 5, 1)\), then \(x \leq y\);

\[
\tilde{x(1)} = (0) \leq_c \tilde{y(1)} = (4), \quad \tilde{x(2)} = (1, 0) \leq_c \tilde{y(2)} = (4, 3), \quad \tilde{x(3)} = (2, 1, 0) \leq_c \tilde{y(3)} = (4, 3, 2), \quad \tilde{x(4)} = (3, 2, 1, 0) \leq_c \tilde{y(4)} = (5, 4, 3, 2), \quad \tilde{x(5)} = (4, 3, 2, 1, 0) \leq_c \tilde{y(5)} = (5, 4, 3, 2, 1).
\]

Next lemma, which is equivalent to the “tableau criterion of the Bruhat-Chevalley ordering on \(S_n\)” [Theorem 2.6.3, [3]] shows that, for two permutations \(x\) and \(y\) of \(S_n\), the inequality \(x \leq y\) can be decided in \(n - 1\) steps. For an alternative proof, see [9].
Lemma 24. For two permutations $x = (a_1, \ldots, a_n)$ and $y = (b_1, \ldots, b_n)$ from $S_n$, $x \leq y$ if and only if $\bar{x}(k) \leq_c \bar{y}(k)$ for $k = 1, \ldots, n - 1$.

We are ready to prove the first half of Theorem 5.

Proposition 25. The union $(R_{n,n-1} \cup R_{n,n}, \leq)$ is isomorphic to the poset $(S_{n+1}, \leq)$.

We depict the isomorphism between $S_4$ and $R_{3,3} \cup R_{3,2}$ in Figure 3.

Proof. Let $u$ and $w$ denote the rook matrices $u = (0, 1, 2, \ldots, n-1)$ and $w = (n, n-1, \ldots, 2, 1)$. The union $R_{n,n-1} \cup R_{n,n}$ forms the interval $[u,w]$. We define a map $\psi$ between $[u,w]$ and $S_{n+1}$ as follows. If $x = (a_1, \ldots, a_n) \in [v,w]$, then

$$\psi(x) = (a_1+1, a_2+1, \ldots, a_n+1, a_x),$$

where $a_x$ is the unique element of the set $[n+1] \setminus \{a_1+1, a_2+1, \ldots, a_n+1\}$.

We have two immediate observations.

1. If $x$ is already a permutation (in $R_{n,n}$), then $a_x = 1$.

2. $\psi$ is injective, hence by Lemma 6, it is bijective as well.

Now, let $x = (a_1, \ldots, a_n)$ and $y = (b_1, \ldots, b_n)$ be two elements in $[u,w]$ such that $x \leq y$. For the sake of brevity, denote the “shifted” sequence $(a_1 + 1, \ldots, a_n + 1)$ associated with $x$ by $x'$. Since increasing each entry of $x$ and $y$ by 1 does not change the relative sizes of the entries of $x$ and $y$, we have

$$x' \leq y'.$$

Recall that this is equivalent to saying that $\bar{x}'(k) \leq_c \bar{y}'(k)$ for all $k = 1, \ldots, n$. Since, $x'$ is the $n$th truncation $\psi(x)(n)$ of the permutation $\psi(x)$, the proof of the theorem is complete by considering Lemma 24. The converse statement “$\psi(x) \leq \psi(y) \implies x \leq y''$ follows from the same argument. Therefore, $\psi$ is a poset isomorphism.

Unfortunately, the map $\psi$ defined in (13) does not restrict to partial involutions nicely enough, so we need another order preserving injection in $P_{n,n-1} \cup P_n$ onto $I_{n+1}$. Let $u = (0, n, n-1, \ldots, 2)$ and let $\iota = (1, 2, \ldots, n)$. Observe that the rank-control matrix of $u$ is the smallest, and that the rank-control matrix of $\iota$ is the largest among
Figure 3: $S_4$ in $(R_3, \leq)$.
all elements of \( P_{n,n-1} \cup P_{n,n} \). Therefore, the union \( P_{n,n-1} \cup P_{n,n} \) is the underlying set of the interval \([\iota, u]\) of \( P_n \).

Let \( x = (a_1, \ldots, a_n) \in [\iota, u] \) be given in one-line notation. There are two possibilities:

1. there is an \( i \in [n] \) such that \( a_i = 0 \), or
2. \( x \) is a permutation.

If \( a_i = 0 \) for some \( i \in [n] \), then we define \( b_i = n + 1 \) and for \( j \in [n] \setminus \{i\} \) we set \( b_j = a_j \). In addition, in this case, we define \( b_{n+1} \) to be the unique element of the set \( \{0, 1, \ldots, n\} - \{a_1, \ldots, a_n\} \). If the second case, we set \( b_j = a_j \) for \( j = 1, \ldots, n \) and define \( b_{n+1} = n + 1 \). Finally, we define \( \phi : [\iota, u] \to I_{n+1} \) by

\[
\phi(x) = (b_1, \ldots, b_{n+1}).
\]

For example,

\[
x = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\Rightarrow \phi(x) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}.
\]

We are ready to prove the second half of Theorem 5.

**Proposition 26.** The union \( (P_{n,n-1} \cup P_{n,n}, \leq) \) is isomorphic to the poset \((I_{n+1}, \leq)\).

We depict the isomorphism between \( I_4 \) and \( P_{3,3} \cup P_{3,2} \) in Figure 4.

**Proof.** Let \( \phi \) be defined as in (14). By its construction, \( \phi \) is injective. Therefore, by Lemma 6, Part 2., it is enough to show that \( \phi \) is order preserving.

Let \( x \) and \( y \) be two elements in \([\iota, u]\) such that \( x \leq y \), hence \( R(y) \leq_R R(x) \). Note that the upper-left \( n \times n \) portion of the rank-control matrix of \( \phi(x) \) is equal to \( R(x) \). The same is true for \( \phi(y) \) and \( R(y) \). Let \( R_{i,j}^{\phi(x)} \) denote the \((i,j)\)th entry of \( R(\phi(x)) \). Since \( \phi(x) \) is a permutation in \( I_{n+1} \), we have

\[
R_{n+1,i}^{\phi(x)} = i \quad \text{and} \quad R_{j,n+1}^{\phi(x)} = j
\]

for all \( i, j \in [n+1] \). The same is true for \( R(\phi(y)) \). Therefore,

\[
R(\phi(y)) \leq_R R(\phi(x))
\]

and the proof is complete. \( \square \)
Figure 4: $I_4$ in $(P_3, \leq)$.
It follows from Propositions 25 and 26 that Theorem 5 is true;
\[ R_{n,n-1} \cup R_{n,n} \cong S_{n+1} \quad \text{and} \quad P_{n,n-1} \cup P_{n,n} \cong I_{n+1}. \]

Finally, we prove Theorem 4, which states that \( R_{n,k} \) and \( P_{n,k} \) are Eulerian if and only if \( k = n \) or \( k = n - 1 \). First of all, \( R_{n,n} \cong S_n \), and by Theorem 5, \( R_{n,n-1} \) is isomorphic to an interval in \( S_{n+1} \). Thus, both \( R_{n,n} \) and \( R_{n,n-1} \) are Eulerian. The same argument is true for both of the posets \( P_{n,n} \) and \( P_{n,n-1} \). Therefore, to finish the proof it is enough to show that, for \( k \neq n, n - 1 \), \( R_{n,k} \) and \( P_{n,k} \) are not Eulerian. To this end, for \( k \leq n - 2 \), let \( v_k, v'_k \) and \( v''_k \) denote the elements
\[
\begin{align*}
v_k &= (0, \ldots, 0, 0, 1, 2, \ldots, k), \\
v'_k &= (0, \ldots, 0, 1, 0, 2, \ldots, k), \\
v''_k &= (0, \ldots, 1, 0, 0, 2, \ldots, k)
\end{align*}
\]
in \( R_{n,k} \). Observe that the interval \([v_k, v''_k] \subset R_{n,k}\) has exactly three elements \( v_k, v'_k, v''_k \), hence it is not Eulerian. Similarly, for \( k \leq n - 2 \), let \( u_k, u'_k \) and \( u''_k \) denote the elements
\[
\begin{align*}
u_k &= (1, 2, \ldots, k, 0, \ldots, 0), \\
u'_k &= (1, 2, \ldots, k - 1, 0, k + 1, 0, \ldots, 0), \\
u''_k &= (1, 2, \ldots, k - 1, 0, 0, k + 2, 0, \ldots, 0)
\end{align*}
\]
in \( I_{n,k} \). In this case, the interval \([u_k, u''_k] \subset P_{n,k}\) has exactly three elements \( u_k, u'_k, u''_k \), and therefore, it is not Eulerian. This finishes the proof of Theorem 4.

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**References**


