

Loop-augmented forests and a variant of the Foulkes' conjecture

Mahir Bilen Can¹ and Jeff Remmel²

¹Tulane University, New Orleans; mahirbilencan@gmail.com

²UC San Diego; jremmel@ucsd.edu

August 6, 2017

Abstract

A loop-augmented forest is a labeled rooted forest with loops on some of its roots. By exploiting an interplay between nilpotent partial functions and labeled rooted forests we investigate the permutation action of the symmetric group on loop-augmented forests. Furthermore, we describe an extension of the Foulkes' conjecture and prove a special case. Among other important outcomes of our analysis are a complete description of the stabilizer subgroup of an idempotent in the semigroup of partial transformations and a generalization of the (Knuth-Sagan) hook length formula.

Keywords: Labeled rooted forests, symmetric group, plethysm.

MSC: 20C30, 05E10, 16W22

1 Introduction

This paper is a natural continuation of our earlier work [1], where we considered a permutation action of the symmetric group on the set of labeled rooted forests. The origin of this representation goes back to the semigroup of partial transformations on a finite set. It turns out, as we will show how in the sequel, the circle of ideas here naturally leads to new variants of a long standing conjecture in representation theory, namely the Foulkes' conjecture. Let us recall this famous conjecture: If $m \leq n$, then the multiplicity of an irreducible representation V of \mathcal{S}_{mn} in the induced trivial representation $\text{Ind}_{\mathcal{S}_m \wr \mathcal{S}_n}^{\mathcal{S}_{mn}} 1$ is less than or equal to the multiplicity of V in $\text{Ind}_{\mathcal{S}_n \wr \mathcal{S}_m}^{\mathcal{S}_{mn}} 1$. We state this conjecture in a combinatorial way, which, in sense, is the precursor of our work.

Consider a set of vertices with mn elements which are placed along a line and grouped into blocks of m elements so that there are n blocks in total. We will distribute the labels

$1, 2, \dots, mn$ on these vertices and call two resulting distributions equivalent if one configuration can be obtained from the other by permuting the labels within the same group and/or by permuting the blocks. For example, the three configurations in the first line of Figure 1, where $n = 3$ and $m = 2$, are in the same equivalence class. (For convenience we separated individual blocks in each configuration by putting a bar between them.) In the second line of Figure 1, we have two inequivalent configurations since the set of elements of the first block of the first configuration is not equal to any set of elements of any blocks in the second configuration.

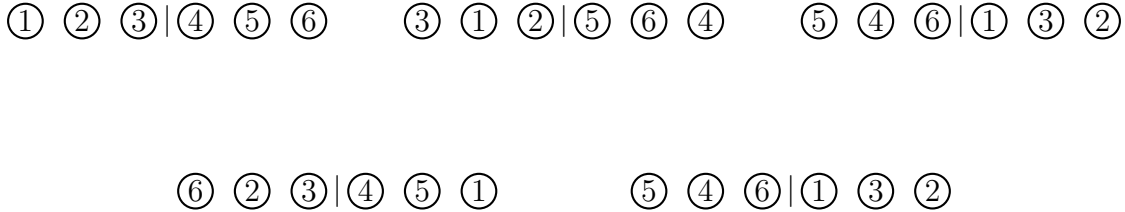


Figure 1: Three equivalent (in the first line), two inequivalent (in the second line) configurations.

Let us denote by o'_1 the vector space spanned by all such equivalence classes. Next, we define another vector space, denoted by o_1 , by changing the roles of m and n . Clearly both of o_1 and o'_1 are \mathcal{S}_{mn} -modules. In this terminology, the Foulkes' conjecture states that the multiplicity of an irreducible \mathcal{S}_{mn} -module V in o'_1 is less than or equal to the multiplicity of V in o_1 .

The conjecture, a special case of which we will prove is the following natural extension of Foulkes' conjecture. Let m and n be positive integers such that $n > m \geq 2$. Consider m copies of the labeled rooted tree on n vertices where the root has exactly $n - 1$ children. See Figure 2, where $n = 4$ and $m = 2$.

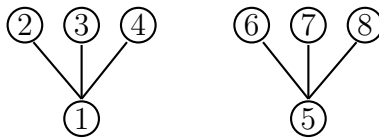


Figure 2: An extension of Foulkes' conjecture, combinatorially.

The symmetric group \mathcal{S}_{mn} acts on vertices without breaking the relations between labels imposed by the edges of the underlying trees. Hence we get a representation denoted by o'_2 . In a similar manner, next, we consider n copies of labeled rooted tree on m vertices where the root has exactly $m - 1$ children. This time the \mathcal{S}_{mn} representation is denoted by o_2 . Now, let $V = V(\lambda)$ be an irreducible representation of \mathcal{S}_{mn} indexed by a partition λ which has more than two parts. Then the multiplicity of V in o_2 is greater than or equal to the multiplicity of V in o'_2 . One of the main results of our paper is the proof of this conjecture when $m = 2$.

As we mentioned before, we arrived at this conjecture by considering nilpotent partial transformations and rook theory. Our goal here is to further develop this topic, so we start with recalling the relevant concepts and results from [1]. This way, we hope that the reader will see a coherent picture of our work and the origins of the conjecture we have above.

Let n denote a positive integer, $[n]$ and $\overline{[n]}$ denote the sets $\{1, \dots, n\}$ and $[n] \cup \{0\}$, respectively. We set

$$\begin{aligned} \mathcal{F}ull_n & : \text{ the full transformation semigroup on } \overline{[n]}; \\ \mathcal{P}_n & : \text{ the semigroup of partial transformations on } [n]; \\ \mathcal{R}_n & : \text{ the rook monoid on } [n]; \\ \mathcal{C}_n & : \text{ the set of nilpotent partial transformations on } [n]. \end{aligned}$$

For us, a partial transformation on $[n]$ is a function $f : A \rightarrow [n]$, where A is a nonempty subset of $[n]$. A *full transformation* on $\overline{[n]}$ is a function $g : \overline{[n]} \rightarrow \overline{[n]}$ and there is an “extension by 0” morphism from partial transformations into full transformations,

$$\begin{aligned} \varphi_0 & : \mathcal{P}_n \rightarrow \mathcal{F}ull_n \\ f & \longmapsto \varphi_0(f) \end{aligned}$$

which is defined by

$$\varphi_0(f)(i) = \begin{cases} f(i) & \text{if } i \text{ is in the domain of } f; \\ 0 & \text{otherwise.} \end{cases}$$

The map φ is an injective semigroup homomorphism.

By a *rook monoid* we mean the monoid of injective partial transformations. We will often refer to the elements of \mathcal{R}_n as rook placements (on an $n \times n$ grid). Since \mathcal{P}_n is embedded into $\mathcal{F}ull_n$ and since the latter monoid contains a zero transformation it makes sense to talk about nilpotent partial transformations in \mathcal{P}_n . In particular, \mathcal{C}_n and $\mathcal{C}_n \cap \mathcal{R}_n$ are well defined although their definitions require an embedding of \mathcal{P}_n into $\mathcal{F}ull_n$. (In literature, many authors assume that $0 \in \mathcal{R}_n$.)

The unit groups of \mathcal{P}_n and \mathcal{R}_n are the same and equal to \mathcal{S}_n . Moreover, the conjugation action of \mathcal{S}_n on itself extends to these semigroups as well as to the subset \mathcal{C}_n of nilpotent partial transformations. Therefore we have \mathcal{S}_n -representations on the vector spaces $\mathbb{C}\mathcal{P}_n$, $\mathbb{C}\mathcal{R}_n$ and on $\mathbb{C}\mathcal{C}_n$.

The combinatorial significance of \mathcal{C}_n , which is crucial to our purposes, stems from the fact that there is a bijection between \mathcal{C}_n and the set of labeled rooted forests on n vertices. Furthermore, the conjugation action of \mathcal{S}_n on functions in \mathcal{C}_n translates to the permutation action of \mathcal{S}_n on the labels. See [1].

Let τ be a rooted forest on n vertices. τ is called labeled if there exists a bijective map ϕ from $[n]$ onto the vertex set of τ . Throughout when we talk about labeled rooted forests, to relax our notation we omit writing the corresponding labeling function despite the fact that the action of \mathcal{S}_n does not change the underlying forest but the labeling function only. In accordance with this convention, when we write $\mathcal{S}_n \cdot \tau$ we actually mean the orbit

$$\mathcal{S}_n \cdot (\tau, \phi) = \{(\tau, \phi') : \phi' = \sigma \cdot \phi, \sigma \in \mathcal{S}_n\}. \quad (1)$$

In [1], the \mathcal{S}_n -representation defined by the orbit (1) is called the odun of τ . It only depends on τ , so we write $\text{Stab}_{\mathcal{S}_n}(\tau)$ to denote the stabilizer subgroup of the pair (τ, ϕ) . As an \mathcal{S}_n -module, the vector space of functions on the right cosets, that is $\mathbb{C}[\mathcal{S}_n/\text{Stab}_{\mathcal{S}_n}(\tau)]$ is isomorphic to the odun of τ , which is equivalent to the induced representation $\text{Ind}_{\text{Stab}_{\mathcal{S}_n}(\tau)}^{\mathcal{S}_n} \mathbf{1}$. It turns out that the Frobenius character of this module is recursive in nature and computable by certain combinatorial rules. To avoid introducing too many notation, we illustrate these rules on an example. Let τ be the rooted forest depicted in Figure 3 and let τ_i , $i = 1, 2, 3$ denote its connected components (from left to right in the figure). Let F_τ and F_{τ_i} , $i = 1, 2, 3$

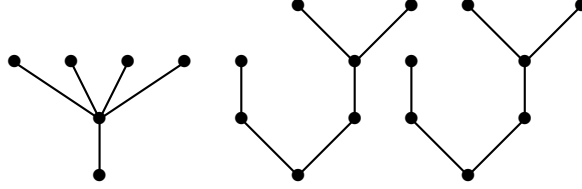


Figure 3: An example.

denote the corresponding Frobenius characters. Since $\tau_1 \neq \tau_2 = \tau_3$, we have

$$F_\tau = F_{\tau_1} \cdot s_2[F_{\tau_2}].$$

Here, s_2 is the Schur function $s_{(2)}$, the bracket stands for the plethysm of symmetric functions and the dot stands for ordinary multiplication. (We will explain plethysm in detail in the sequel.) More generally, if a connected component τ' , which of course is a rooted tree appears k -times in a forest τ , then F_τ has $s_k[F_{\tau'}]$ as a factor. Now we proceed to explain the computation of the Frobenius character of a rooted tree. As an example we use F_{τ_1} of Figure 3. The combinatorial rule that we obtained in [1] is simple; it is the removal of the root from tree. The effect on the Frobenius character of this simple rule is as follows: Let τ'_1 denote the rooted forest that we obtain from τ_1 by removing the root. Then $F_{\tau_1} = s_1 \cdot F_{\tau'_1}$. Thus, by the repeated application of this rule and the previous factorization rule, we obtain

$$F_{\tau_1} = s_1 \cdot F_{\tau_1} = s_1 \cdot s_1 \cdot s_4[s_1].$$

We compute F_{τ_2} by the same method and we arrive at the following satisfactory expression for the Frobenius character of τ :

$$F_\tau = s_1^2 \cdot s_4[s_1] \cdot F_{\tau_2} = s_1^2 \cdot s_4 \cdot s_2[s_1^5 \cdot s_2].$$

Let τ be a rooted forest, a be a vertex, and denote by τ_a^0 the rooted subforest whose set of vertices is $\{b \in \tau : b < a\}$. We set τ_a to denote the rooted subtree whose set of vertices is $\{b \in \tau : b \leq a\}$. Finally, let τ_a denote the set of children $\{a_1, \dots, a_r\}$ of τ_a such that the corresponding subtrees τ_{a_i} ($i = 1, \dots, r$) are mutually distinct. Finally, let $m(a; a_i) = m_\tau(a; a_i)$ ($i = 1, \dots, r$) denote the number of copies of τ_{a_i} that appear in τ_a^0 . In this notation, the dimension of the odun of τ turns out to be

$$\dim_{\mathbb{C}} o(\tau) = \frac{n!}{\prod_{a \in \tau} \prod_{b \in \tau_a} m(a; b)!}, \quad (2)$$

where n is the number of vertices of τ . See [1, Theorem 9.3]. This is a generalization of the Knuth-Sagan hook length formula for the rooted trees.

Another important advancement of [1] is the determination of the rooted trees whose odun afford the sign representation. We will give a detailed explanation of the arguments involved with this computation in the sequel. Let us mention in passing that the generating function for the number of rooted forests which afford the sign representation is $\sum_{n \geq 2} 2^{n-2} x^n$. Here, n is interpreted as the number of vertices.

We are now ready to introduce our new combinatorial idea by means of which we will generalize the results we mentioned before. The algebraic significance of these objects will become clear in the sequel. A *loop-augmented forest* is a rooted forest with loops on some of its vertices. See, for example, Figure 4, where we depict a loop-augmented forest on 22 vertices and four loops. It is a well known variation of the Cayley's theorem that the

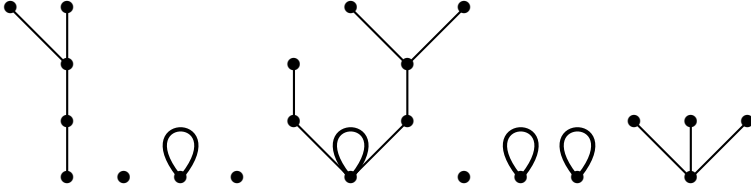


Figure 4: A loop-augmented forest.

number of labeled forests on n vertices with k roots is equal to $\binom{n-1}{k-1} n^{n-k}$. See [3], Theorem D, pg 70. It follows that the number of loop-augmented forests on n vertices with k roots is $2^k \binom{n-1}{k-1} n^{n-k}$. It follows from generating function manipulations that the number of loop-augmented forests on n vertices for $n \geq 2$ is $2n^{n-3}$.

Let τ be a rooted tree on n vertices. We group the maximal branches of τ according to the number of vertices on each maximal branch. As a convention, let us collect the maximal branches with more vertices on the left hand side of the graph and repeat this procedure inductively for every maximal branch and for their maximal branches, and so on. We do not worry about the ordering of the maximal branches in the same group. As an example, see Figure 5. Once the grouping is done, we label the vertices by $[n]$ from left to right and from bottom to top, decreasing the label by 1 each time. Here, n is the number of vertices and the root of τ is labeled by n . Notwithstanding the ambiguity created by not ordering

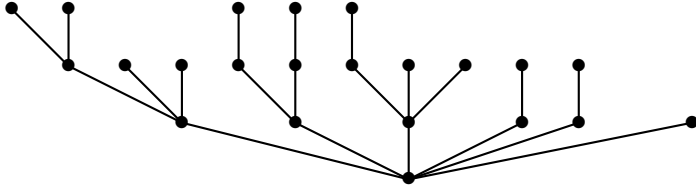


Figure 5: A grouping of the branches of a rooted tree.

the maximal branches of the same size, the resulting incidence matrix from our labeling is a

strictly upper triangular (nilpotent) matrix. We adopt this procedure to the rooted forests by first grouping (and depicting left to right) the rooted trees with respect to the number of vertices they have. The resulting incidence matrix of the forest is a direct sum of nilpotent matrices, hence it is nilpotent also. Notice that this procedure also explains why there is a bijection between labeled rooted forests and nilpotent partial functions; for $\tau \in \mathcal{C}_n$ with respect to any labeling that we just described let f be the partial function defined by $f(i) = j$ whenever i is a child of j in τ and there is an edge between i and j . Then f is nilpotent. In the light of this fact, from now on we will not distinguish between the notation for a nilpotent partial function and the corresponding labeled rooted forest; we will typically use the Greek letter τ to represent both of them.

We proceed to explain how to interpret loop-augmented forests in terms of partial functions. Let σ be a loop-augmented forest. Then some of the roots of σ have loops. There is still a partial function for σ , as defined in the previous paragraph for a rooted forest. The loops in this case correspond to the “fixed points” of the associated function. Indeed, with respect to labeling conventions of the previous paragraph the incidence matrix of σ is upper triangular however it has both of 1’s and 0’s on its main diagonal. The resulting matrix can be viewed as the matrix representation of the partial function of σ . (*Warning:* Unlike our notation in this paragraph, in the sequel we will often use lower case Latin letters for denoting both of loop-augmented forests and partial functions.) We remark here that the correspondence between loop-augmented forests and (certain) partial functions fails if we consider adding loops on non-roots; the resulting objects simply do not give partial functions but to, what should logically be called the “partial relations.”

The subset of \mathcal{P}_n which corresponds to the loop-augmented forests will be defined in the next section. Now we move on to explain the underlying algebraic theme of our work. Recall that the permutation action of \mathcal{S}_n on the labels translates to the conjugation action on the incidence matrices. Clearly, the conjugates of a nilpotent (respectively unipotent) matrix are still nilpotent (respectively unipotent). Let U be an arbitrary upper triangular matrix. Then U is equal to a sum of the form $D + N$, where D is a diagonal matrix, N is a nilpotent matrix. If σ is a permutation matrix of the same size as U , then we conclude from the equalities $\sigma \cdot U = \sigma U \sigma^{-1} = \sigma D \sigma^{-1} + \sigma N \sigma^{-1}$ that the conjugation action on loop-augmented forests is equivalent to the simultaneous conjugation action on labeled rooted forests and the representation of \mathcal{S}_n on diagonal matrices.

We will now give a quick overview of the main results of our paper and describe its organization. In Section 2 we review the most basic facts about the idempotents of the semi-groups $\mathcal{F}ull_n, \mathcal{P}_n$, and \mathcal{R}_n . In addition we review the basic theory plethysm for symmetric functions. In Section 3 we prove our first main result, Theorem 3.1, which states that the stabilizer subgroup in \mathcal{S}_n of a loop-augmented forest (viewed as a partial function on $[n]$) has a direct product decomposition of the form $Z(\sigma) \times \text{Stab}_{\mathcal{S}_{n-k}}(\tau)$, where $Z(\sigma)$ is the centralizer of a permutation σ in \mathcal{S}_k and $\text{Stab}_{\mathcal{S}_{n-k}}(\tau)$ is the stabilizer subgroup in \mathcal{S}_{n-k} of a labeled rooted forest τ . It follows from this result that the Frobenius character of the \mathcal{S}_n orbit of a loop-augmented forest f is of the form $s_\lambda \prod_{i=1}^r h_{\lambda_i} [F_{\tau_i}]$, where τ_1, \dots, τ_r are the distinct connected components of τ where τ_i occurs λ_i times. We record this theorem as Theorem 4.1 in

Section 4. Another consequence of the factorization of the stabilizer subgroup is a dimension formula for the \mathcal{S}_n representation defined by f (Corollary 3.1). If $(1^{m_1} 2^{m_2} \dots k^{m_k})$ denotes the conjugacy type of σ in \mathcal{S}_k , then the dimension of $\mathbb{C}[\mathcal{S}_n \cdot f]$ is given by

$$\frac{n!}{k!} \frac{\prod_j j^{m_j} (m_j!)}{(\prod_{a \in \tau} \prod_{b \in \gamma_a} m(a; b)!)}, \quad (3)$$

where $m(a; b) = m_\tau(a; b)$ is as defined in (2). The goal of Section 5 is to determine the loop-augmented forests that afford a sign representation. We prove in Theorem 5.1 that the number of loop-augmented forests on n vertices with k loops whose \mathcal{S}_n orbit admits a sign representation is 2^{n-k-2} . In particular, the total number of loop-augmented forests on n vertices admitting a sign representation is $2^{n-1} - 1$. In the subsequent Section 6 we prove the stand alone algebraic result of our paper: If $e \in \mathcal{P}_n$ is an arbitrary idempotent, then there exists a multiset of integers $\{\beta_1^{m_1}, \dots, \beta_s^{m_s}\}$ and $m \in \mathbb{N}$ such that

1. $m + \sum_{i=1}^s m_i \beta_i = n$;
2. $\text{Stab}_{\mathcal{S}_n}(e) \simeq (\mathcal{S}_{m_1} \wr \mathcal{S}_{\beta_1}) \times (\mathcal{S}_{m_2} \wr \mathcal{S}_{\beta_2}) \times \dots \times (\mathcal{S}_{m_s} \wr \mathcal{S}_{\beta_s}) \times \mathcal{S}_m$.

We record this result in Corollary 6.2. Finally, in Section 7, we prove the special case of our aforementioned conjecture by using symmetric functions. (See Conjecture 7.1 for an equivalent statement in terms of symmetric functions and plethysm.)

2 Preliminaries

Structures of the semigroups \mathcal{P}_n and \mathcal{R}_n become more transparent by virtue of matrices; each element g of \mathcal{P}_n has a unique $n \times n$ matrix representative $g = (g_{ij})_{i,j=1}^n$ defined by

$$g_{ij} = \begin{cases} 1 & \text{if } g(i) = j; \\ 0 & \text{otherwise} \end{cases} \quad (i, j \in [n]).$$

What is important for us is that the matrix product translates to the semigroup multiplication (compositions). In particular, a partial transformation $f \in \mathcal{P}_n$ is nilpotent if some power of the corresponding matrix $f = (f_{i,j})_{i,j=1}^n$ is the zero matrix $\mathbf{0}_n$.

2.1 Idempotents.

An idempotent in a semigroup S is an element $e \in S$ satisfying $e^2 = e$. The identity transformation $id_n : [n] \rightarrow [n]$ of the unit group is an idempotent, and conversely for any idempotent $e \in S := \mathcal{P}_n$ there is a corresponding ‘‘local group’’ S_e determined by e ; it is defined as $S_e := e\mathcal{P}_n e$. The identity of the local group S_e is e . We denote the idempotent set of a semigroup S by $E(S)$.

By a *diagonal idempotent* in \mathcal{P}_n we mean an idempotent of \mathcal{R}_n . The nomenclature follows from the fact that the matrix representation of $e \in E(\mathcal{R}_n)$ is a diagonal matrix. $E(\mathcal{R}_n) \subset \mathcal{R}_n$

has exactly 2^n elements (including the 0-matrix) and it forms a commutative submonoid of \mathcal{R}_n . It is not difficult to see that $E(\mathcal{R}_n)$ is isomorphic to the “face lattice” of the regular “ n -simplex” in \mathbb{R}^n . For example,

$$E(\mathcal{R}_2) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

Let U denote the unit group of a semigroup S . For our purposes it is important to note that $E(S)$ is closed under the conjugation action

$$g \cdot e = geg^{-1}, \quad g \in U, e \in E(S).$$

In particular we see that $E(\mathcal{R}_n)$ is an \mathcal{S}_n -set that is contained in $E(\mathcal{P}_n)$. For example,

$$E(\mathcal{P}_2) = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\} \cup E(\mathcal{R}_2).$$

We list some easy facts about the set of idempotents of \mathcal{P}_n . For the proofs of some of these facts (and more) we recommend [4].

1. The number of idempotents in \mathcal{P}_n is given by $\sum_{k=0}^n \binom{n}{k} (k+1)^{n-k}$.
2. The number of idempotents in $\mathcal{F}ull_n$ is given by $\sum_{k=1}^n \binom{n}{k} k^{n-k}$.
3. The idempotents in \mathcal{P}_n do *not* form a subsemigroup of \mathcal{P}_n for $n \geq 2$.
4. The idempotents in $\mathcal{F}ull_n$ do *not* form a subsemigroup of $\mathcal{F}ull_n$ for $n \geq 3$.
5. The idempotents in \mathcal{R}_n form a submonoid of \mathcal{R}_n .

Let e be a diagonal idempotent in \mathcal{P}_n . Since all idempotents of rank k in \mathcal{R}_n are related to each other via conjugation, the \mathcal{S}_n orbit of e is uniquely determined by the number $k = \text{rank}(e)$. We define

$$\mathcal{N}(n, k) = \{f \in \mathcal{P}_n : f^m \text{ is a rank } k \text{ diagonal idempotent for some } m \geq 1\}, \quad (4)$$

which is closed under conjugation action of \mathcal{S}_n . Clearly, if $k = 0$, then $\mathcal{N}(n, k)$ is equal to \mathcal{C}_n , the set of nilpotent elements in \mathcal{P}_n .

Remark 2.1. *By relaxing the requirement on diagonal property, we could have considered*

$$\{f \in \mathcal{P}_n : f^m \text{ is a rank } k \text{ idempotent for some } m \geq 1\}. \quad (5)$$

This set properly contains $\mathcal{N}(n, k)$. However, the matrix rank of idempotents is not a refined invariant with respect to conjugation action. To justify this claim, we consider

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad e_2 := \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

These idempotents are of the same rank but they are not conjugate. To capture differences between the conjugacy classes of e_1 and e_2 , we will introduce below the concept of “standard idempotent.”

Let $e \in E(\mathcal{P}_n)$ be an idempotent with the property that the only nonzero row of e has 1's as its entries. For example, $e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. Such idempotents, viewed as functions, are constant. Conversely, any nonzero constant function $f : [n] \rightarrow [n]$ defines an idempotent $e \in \mathcal{P}_n$ with a single row of 1's.

Definition 2.1. *An idempotent is called constant, and denoted by $c^{(n)}$, if the corresponding function is 1. In other words $c^{(n)}(i) = 1$ for all $i \in \{1, \dots, n\}$. An idempotent in \mathcal{P}_n is called standard if it is a direct sum of constant idempotents of various sizes plus a 0-matrix.*

Let $x \in \mathcal{P}_n$. The following set (not to be confused with $\mathcal{N}(n, k)$, where k is a number) will play an important role in the sequel:

$$\mathcal{N}(n, x) := \{f \in \mathcal{P}_n : f^m \text{ is conjugate to } x \text{ for some } m \geq 1\}. \quad (6)$$

Lemma 2.1. *If $x = e$ be an idempotent, then $\mathcal{N}(n, e)$ is closed under conjugation. Furthermore, if e is a diagonal idempotent of rank k , then $\mathcal{N}(n, k) = \mathcal{N}(n, e)$.*

Proof. Straightforward. □

Remark 2.2. *The reason for we introduce the set (4) is because $\mathcal{N}(n, k)$ is precisely the subset of \mathcal{P}_n that corresponds to the loop-augmented forests. Thus, our main goal is to understand the \mathcal{S}_n -module structure on $\mathcal{N}(n, k)$.*

2.2 Symmetric functions and plethysm.

In representation theory, plethysm refers to a general idea on a functorial operation, which is defined from a representation to another one. More concretely, it is a description of the decompositions, with multiplicities, of representations derived from a given representation V such as $V \otimes V, \wedge^k(V), \wedge^k(\wedge^l(V)), \text{Sym}^k(V), \text{Sym}^l(\text{Sym}^k(V)), V^*$.. In practice one often considers functors attached to the weight of an irreducible representation such as Schur functors. Let \mathbb{S}_μ resp. \mathbb{S}_λ denote the Schur functors indexed by the partitions μ and λ , respectively.. Then $\mathbb{S}_\lambda \circ \mathbb{S}_\mu := \mathbb{S}_\lambda(\mathbb{S}_\mu(\mathbb{C}^n))$ is called the plethysm of \mathbb{S}_λ with \mathbb{S}_μ . There is a corresponding operation on the ring of symmetric functions.

The k -th power-sum symmetric function, denoted by p_k , is the sum of k -th powers of the variables. For a partition $\lambda = (\lambda_1 \leq \dots \leq \lambda_l)$, we define p_λ to be equal to $\prod_i p_{\lambda_i}$. (We will maintain the convention of writing parts of a partition in the increasing order throughout our paper.) The k -th complete symmetric function, h_k is defined to be the sum of all monomials $x_{i_1}^{a_1} \dots x_{i_r}^{a_r}$ with $\sum a_i = k$ and h_λ is defined to be equal to $\prod_{i=1}^l h_{\lambda_i}$. The Schur function associated with λ , denoted by s_λ , is the symmetric function defined by the determinant $\det(h_{\lambda_i+j-i})_{i,j=1}^l$. The power sum and Schur symmetric functions will play special roles in our computations via the Frobenius character map

F : class functions on $\mathcal{S}_n \rightarrow$ symmetric functions

$$\delta_\sigma \mapsto \frac{1}{n!} p_\lambda,$$

where $\sigma \subset \mathcal{S}_n$ is a conjugacy class of type λ and δ_σ is the indicator function

$$\delta_\sigma(x) = \begin{cases} 1 & \text{if } x \in \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

It turns out that if χ^λ is the irreducible character of \mathcal{S}_n indexed by the partition λ , then $\mathbf{F}(\chi^\lambda) = s_\lambda$. In the sequel, we will not distinguish between representations of \mathcal{S}_n the corresponding characters. In particular, we will often write the *Frobenius character of an orbit* to mean the image under \mathbf{F} of the character of the representation of \mathcal{S}_n that is defined by the action on the orbit.

For two irreducible characters χ^μ and χ^λ indexed by partitions λ and μ , the Frobenius characteristic image of the plethysm $\chi^\lambda[\chi^\mu]$ is the plethystic substitution $s_\lambda[s_\mu]$ of the corresponding Schur functions. Roughly speaking, the plethysm of the Schur function s_λ with s_μ is the symmetric function obtained from s_λ by substituting the monomials of s_μ for the variables of s_λ . In the notation of [6]; the plethysm of symmetric functions is the unique map $[\cdot] : \Lambda \times \Lambda \rightarrow \Lambda$ satisfying the following three axioms:

P1. For all $m, n \geq 1$, $p_m[p_n] = p_{mn}$.

P2. For all $m \geq 1$, the map $g \mapsto p_m[g]$, $g \in \Lambda$ defines a \mathbb{Q} -algebra homomorphism on Λ .

P3. For all $g \in \Lambda$, the map $h \mapsto h[g]$, $h \in \Lambda$ defines a \mathbb{Q} -algebra homomorphism on Λ .

Although the problem of computing the plethysm of two (arbitrary) symmetric functions is very difficult, there are some useful formulas for Schur functions:

$$s_\lambda[g + h] = \sum_{\mu, \nu} c_{\mu, \nu}^\lambda (s_\mu[g])(s_\nu[h]), \quad (7)$$

and

$$s_\lambda[gh] = \sum_{\mu, \nu} \gamma_{\mu, \nu}^\lambda (s_\mu[g])(s_\nu[h]). \quad (8)$$

Here, g and h are arbitrary symmetric functions, $c_{\mu, \nu}^\lambda$ is a scalar, and $\gamma_{\mu, \nu}^\lambda$ is $\frac{1}{n!} \langle \chi^\lambda, \chi^\mu \chi^\nu \rangle$, where the pairing stands for the standard Hall inner product on characters.

In (7) the summation is over all pairs of partitions $\mu, \nu \subset \lambda$, and the summation in (8) is over all pairs of partitions μ, ν such that $|\mu| = |\nu| = |\lambda|$. In the special case when $\lambda = (n)$, or (1^n) we have

$$s_{(n)}[gh] = \sum_{\lambda \vdash n} (s_\lambda[g])(s_\lambda[h]), \quad (9)$$

$$s_{(1^n)}[gh] = \sum_{\lambda \vdash n} (s_\lambda[g])(s_{\lambda'}[h]), \quad (10)$$

where λ' denotes the conjugate of λ .

Finally, let us mention the conjugation property of plethysm:

$$\langle s_\lambda[s_\mu], s_\gamma \rangle = \begin{cases} \langle s_\lambda[s_{\mu'}], s_{\gamma'} \rangle & \text{if } |\mu| \text{ is even,} \\ \langle s_{\lambda'}[s_{\mu'}], s_{\gamma'} \rangle & \text{if } |\mu| \text{ is odd.} \end{cases} \quad (11)$$

3 Nilpotents and diagonal idempotents

Let f be an element of $\mathcal{N}(n, k)$, m be a positive integer such that f^m is a diagonal idempotent in \mathcal{P}_n . Without loss of generality we assume that f^m is of the form

$$f^m = \begin{pmatrix} id_k & 0 \\ 0 & 0 \end{pmatrix}. \quad (12)$$

Our first result is about the actual “shape” of this matrix representation of f .

Proposition 3.1. *If f is a partial transformation as in the previous paragraph, then there exist a permutation $\sigma \in \mathcal{S}_k$ and a nilpotent partial transformation τ from \mathcal{C}_{n-k} such that*

$$f = \begin{pmatrix} \sigma & 0 \\ 0 & \tau \end{pmatrix}. \quad (13)$$

Proof. First, we are going to show that f has the block-diagonal form as in (13). We will prove this by contradiction, so we assume that there exists $j \in \{k, \dots, n\}$ such that $f(j) = i < k$. On one hand, since f^m is as in (12), we have $f^m(i) = i$. On the other hand, by our assumption $0 = f^m(j) = f^{m-1}(f(j)) = f^{m-1}(i)$. Therefore, $f^m(i) = 0$, but this is a contradiction. Therefore, f is as in (13) for some $\sigma \in \mathcal{P}_k$ and $\tau \in \mathcal{C}_{n-k}$.

Since the m -th power of $\begin{pmatrix} \sigma & 0 \\ 0 & \tau \end{pmatrix}$ is as in (12), the m -th power of σ is the $k \times k$ identity matrix and the m -th power of τ is $\mathbf{0}_{n-k}$. In other words, σ is a permutation matrix, τ is a nilpotent matrix. The proof is complete. \square

Theorem 3.1. *Let f be a partial transformation of the form $f = \begin{pmatrix} \sigma & 0 \\ 0 & \tau \end{pmatrix}$, where $\sigma \in \mathcal{S}_k$ is a permutation and $\tau \in \mathcal{C}_{n-k}$ is a nilpotent partial transformation. In this case, the stabilizer subgroup in \mathcal{S}_n of f has the following decomposition:*

$$Stab_{\mathcal{S}_n}(f) = Z(\sigma) \times Stab_{\mathcal{S}_{n-k}}(\tau),$$

where $Z(\sigma)$ is the centralizer of σ in \mathcal{S}_k and $Stab_{\mathcal{S}_{n-k}}(\tau)$ is the stabilizer subgroup of τ in \mathcal{S}_{n-k} .

Proof. Let $\pi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{S}_n$ be a permutation matrix given in block form so that A is a $k \times k$ -, D is $n - k \times n - k$ -, B is $k \times n - k$ -, and C is $n - k \times k$ -matrix. Assume that $\pi f \pi^{-1} = f$. We claim that both B and C are 0 matrices. To see this, we use $\pi f = f \pi$ to get four matrix equations:

$$A\sigma = \sigma A \quad (14)$$

$$BN = \sigma B \quad (15)$$

$$C\sigma = NC \quad (16)$$

$$ND = DN. \quad (17)$$

We are going to show only that $B = 0$ since for C the idea of the proof is the same.

First of all, if $N = 0$, then it immediately follows from the above equations that $B = 0$, so we assume that N is a nonzero nilpotent matrix. Let r be the smallest integer such that $BN^r = 0$. If $r = 1$, then since σ is invertible, it follows from equation (15) that $B = 0$, so we continue with the assumption that $r > 1$. In this case, by multiplying equation (15) with N^{r-1} on the right we obtain $\sigma BN^{r-1} = BN^r = 0$. By the same token, since σ is invertible, we see that $BN^{r-1} = 0$. But this contradicts with our assumption on the minimality of r . Therefore, $r = 1$, hence $B = 0$. Since both of B and C are 0, the proof follows from equations (14) and (17). \square

By Proposition 3.1 and Remark 2.2, we see that a partial transformation $f \in \mathcal{P}_n$ represents a loop-augmented forest, then its matrix form is as in (13). By this notation, as an application of Theorem 3.1, we will now record a representation theoretic generalization of the famous Knuth-Sagan hook length formula.

Corollary 3.1. *Let f be partial transformation representing a loop-augmented forest as in (13). If $(1^{m_1} 2^{m_2} \dots k^{m_k})$ is the conjugacy type of the permutation σ , then the dimension of $o(f)$, the \mathcal{S}_n representation defined on the orbit of f is equal to*

$$\frac{n!}{k!} \frac{\prod_j j^{m_j} (m_j!)}{(\prod_{a \in \tau} \prod_{b \in \gamma_a} m(a; b))!},$$

where $m(a; b) = m_\tau(a; b)$ is as defined in (2).

Proof. By Theorem 3.1 we know that the \mathcal{S}_n representation defined by f is isomorphic to $\mathbb{C}[\mathcal{S}_n/Z(\sigma) \times \text{Stab}_{\mathcal{S}_{n-k}}(\tau)]$, hence its dimension is given by $n!/(|Z(\sigma)||\text{Stab}_{\mathcal{S}_{n-k}}(\tau)|)$. We express this number in a different way:

$$\begin{aligned} \dim_{\mathbb{C}} \mathcal{S}_n \cdot f &= \frac{n!}{|Z(\sigma)||\text{Stab}_{\mathcal{S}_{n-k}}(\tau)|} \\ &= \binom{n}{k} \frac{k!}{|Z(\sigma)|} \frac{(n-k)!}{|\text{Stab}_{\mathcal{S}_{n-k}}(\tau)|}. \end{aligned} \tag{18}$$

Note that $|Z(\sigma)|$ is equal to $k!/|C(\sigma)|$, where $C(\sigma)$ is the conjugacy class of σ in \mathcal{S}_k . The cardinality of $C(\sigma)$ is equal to $k!/\prod_j j^{m_j} (m_j!)$. Thus, the rest of the proof follows from (2) and (18). \square

4 Characters

Let f be a partial transformation in $\mathcal{N}(n, k)$. As before, let $o(f)$ denote the corresponding \mathcal{S}_n -representation on the orbit of f . Let $H(f)$ denote the stabilizer subgroup $\text{Stab}_{\mathcal{S}_n}(f)$. As a consequence of Theorem 3.1 we see that $o(f)$ is \mathcal{S}_n -equivariantly isomorphic to the vector space

$$o(f) \simeq \mathbb{C}[\mathcal{S}_n/H(f)], \tag{19}$$

where $H(f) \simeq Z(\sigma) \times \text{Stab}_{\mathcal{S}_{n-k}}(\tau)$ for some permutation $\sigma \in \mathcal{S}_k$ and a nilpotent partial transformation $\tau \in \mathcal{C}_{n-k}$. The \mathcal{S}_n module structure on $\mathbb{C}[\mathcal{S}_n/H(f)]$ is given by the left translation action of \mathcal{S}_n on the cosets of $H(f)$. In other words, $o(f)$ is isomorphic to the induced trivial representation from $H(f)$ to \mathcal{S}_n . Since $H(f)$ decomposes as a direct product, the character of $o(f)$ is equal to the product of characters of the induced trivial representations on the corresponding ambient symmetric subgroups. Stated more precisely, we have the following isomorphism:

$$\text{Ind}_{Z(\sigma) \times \text{Stab}_{\mathcal{S}_{n-k}}(\tau)}^{\mathcal{S}_n} \mathbf{1} \simeq \text{Ind}_{Z(\sigma)}^{\mathcal{S}_k} \mathbf{1} \otimes \text{Ind}_{\text{Stab}_{\mathcal{S}_{n-k}}(\tau)}^{\mathcal{S}_{n-k}} \mathbf{1}. \quad (20)$$

Notation : Let $H \subset G$ be a subgroup. We denote by $\chi_{H,G}$ the character of the induced trivial representation $\text{Ind}_H^G \mathbf{1}_H$. If τ is a loop-augmented forest on m vertices, then we denote the character of the induced trivial representation from $\text{Stab}_{\mathcal{S}_m}(\tau)$ to \mathcal{S}_m by $\chi_{o(\tau)}$.

Lemma 4.1. *We preserve our notation from the previous paragraph and let $f = \begin{pmatrix} \sigma & 0 \\ 0 & \tau \end{pmatrix}$ denote a partial function representing a loop-augmented forest. In this case the character of $o(f)$ has the following factorization:*

$$\chi_{o(f)} = \chi_{Z(\sigma), \mathcal{S}_k} \cdot \chi_{o(\tau)}.$$

Proof. Straightforward. □

The set of cosets of the subgroup $Z(\sigma)$ in \mathcal{S}_k is in natural bijection with the set of conjugates of $\sigma \in \mathcal{S}_k$. In other words, viewed as an \mathcal{S}_k -set, $\mathcal{S}_k/Z(\sigma)$ is identified with the conjugacy class of σ in \mathcal{S}_k . It is well known that the conjugacy classes (of \mathcal{S}_k) are parametrized by the partitions (of k), and a representation on a conjugacy class is irreducible. By the Frobenius characteristic map the irreducible representation on a conjugacy class that is indexed by a partition ν of k is mapped to the Schur function s_ν . Clearly, this argument takes care of the character of the induced representation $\text{Ind}_{Z(\sigma)}^{\mathcal{S}_k} \mathbf{1}$ in (20). For the other tensor factor, we refer to our earlier work: [1, Corollary 6.5] states that if $\lambda = (\lambda_1, \dots, \lambda_r)$ is a partition of $n - k$ and τ is a rooted forest consisting of λ_1 copies of a rooted tree τ_1 , λ_2 copies of a rooted tree τ_2 and so on, then the character of $o(\tau)$ is given by

$$\chi_{o(\tau)} = (\chi^{(\lambda_1)}[\chi_{o(\tau_1)}]) \cdot (\chi^{(\lambda_2)}[\chi_{o(\tau_2)}]) \cdots (\chi^{(\lambda_r)}[\chi_{o(\tau_r)}]). \quad (21)$$

Theorem 4.1. *Let f be a loop-augmented forest on n vertices, then $f \sim \begin{pmatrix} \sigma & 0 \\ 0 & \tau \end{pmatrix}$ for some $\sigma \in \mathcal{S}_k$ and $\tau \in \mathcal{N}(\mathcal{P}_{n-k}; 0)$. Furthermore, if $\nu \vdash k$ is the conjugacy type of σ in \mathcal{S}_k and if the underlying rooted forest of the nilpotent partial transformation τ has λ_1 copies of the rooted tree τ_1 , λ_2 copies of the rooted forest τ_2 and so on, then the character of $o(f)$ is given by*

$$\chi_{o(f)} = \chi^\nu \cdot \chi_{o(\tau)} = \chi^\nu \cdot (\chi^{(\lambda_1)}[\chi_{o(\tau_1)}]) \cdot (\chi^{(\lambda_2)}[\chi_{o(\tau_2)}]) \cdots (\chi^{(\lambda_r)}[\chi_{o(\tau_r)}]).$$

Proof. The proof follows from the discussion in the previous paragraph and Lemma 4.1. □

5 Sign representation

In [1, Section 7] we found a characterization of labeled rooted forests that afford the sign representation.

Definition 5.1. A subtree a of a rooted tree is called a *terminal branch* (TB for short) if any vertex of a has at most 1 successor. A *maximal terminal branch* (or, MTB for short) is a terminal branch that is not a subtree of any terminal branch other than itself. The length (or height) of a TB is the number of vertices it has. We call a rooted tree *blossoming* if all of its MTB's are of even length, or no two odd length MTB's of the same length are connected to the same parent. We call a non-blossoming tree *dry*. Finally, a rooted forest is called *blossoming* if the rooted tree that is obtained by adding a new root to the forest is a blossoming tree; otherwise, the forest is called *dry*.

Note that if a forest is blossoming, then all of its connected components are blossoming. Also, if a single connected component is dry, then the whole forest is dry. In Figure 6 we have the list of all blossoming forests up to 5 vertices.

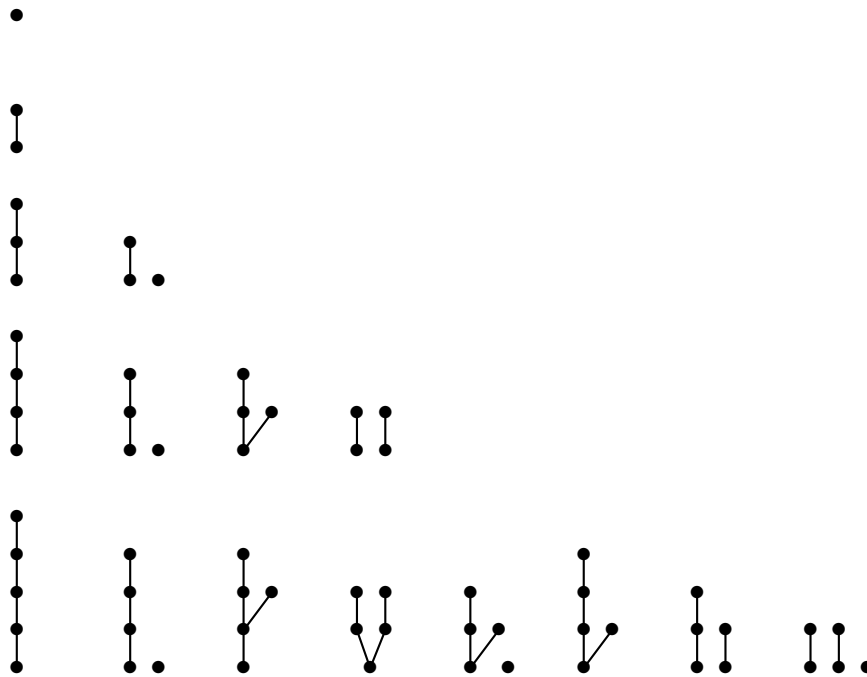


Figure 6: Blossoming forests on $1 \leq n \leq 5$ vertices.

Lemma 5.1 ([1], Proposition 7.9). Let λ be a partition with $|\lambda| = l > 1$ and let τ be a rooted forest on m vertices.

1. If τ is a blossoming forest, then $\langle s_\lambda[F_\tau], s_{(1^l)} \rangle = \begin{cases} 1 & \text{if } \lambda = (1^l); \\ 0 & \text{otherwise.} \end{cases}$

2. If τ is a dry forest, then $\langle s_\lambda[F_\tau], s_{(1^m)} \rangle = 0$.

The following consequence of the above lemma is not very difficult to see.

Corollary 5.1 ([1], Theorem 7.11). *The number of blossoming forests on n vertices is 2^{n-2} .*

We are going to combine these results with our master character formula from Theorem 4.1. Let f be a loop-augmented forest on n vertices as in Theorem 4.1, that is, the partial transformation corresponding to f is \mathcal{S}_n -conjugate to a block diagonal matrix of the form $\begin{pmatrix} \sigma & 0 \\ 0 & \tau \end{pmatrix}$ for some $\sigma \in \mathcal{S}_k$ and $\tau \in \mathcal{N}(n-k; 0)$; $\nu \vdash k$ denotes the conjugacy type of σ in \mathcal{S}_k . It follows from the conclusion of Theorem 4.1 and Lemma 5.1 that

$$\langle \chi_{(1^n)}, \chi_{o(f)} \rangle = \begin{cases} 1 & \text{if } \nu = (1^k) \text{ and } o(\tau) \text{ is the odun of a blossoming forest;} \\ 0 & \text{otherwise.} \end{cases} \quad (22)$$

Theorem 5.1. *The number of loop-augmented forests on n vertices with k loops whose odun affords the sign representation is 2^{n-k-2} . The total number of loop-augmented forests on n vertices admitting the sign representation is $2^{n-1} - 1$.*

Proof. The proof of the first claim follows from Corollary 5.1 and the formula (22). To prove the second claim we first note that k can be any number from 2 to n . Since $1+2+\dots+2^{n-2} = 2^{n-1} - 1$, the proof is finished. \square

6 Non-diagonal idempotents

Let e denote the i -th constant idempotent $c^{(i)}$. Observe that e stays unchanged by multiplication by permutations on the right and therefore the conjugation action on e is given by the left multiplication action; this gives the *standard representation* of \mathcal{S}_n on \mathbb{C}^n . In particular, the stabilizer subgroup of e is isomorphic to $\mathcal{S}_1 \times \mathcal{S}_{n-1} \simeq \mathcal{S}_{n-1}$.

We now assume that e is an idempotent (in \mathcal{P}_n) with k 1's ($0 < k \leq n$) in its i -th row and 0's elsewhere. Then there exists $\sigma \in \mathcal{S}_n$ such that the nonzero entries of $\sigma e \sigma^{-1}$ are in the first row and in the first k columns. In this case, we call $\sigma e \sigma^{-1}$ the standard form of e . (A

constant idempotent $c^{(r)}$ is already in its standard form.) For example, if $e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$,

then the standard form of e is given by $\sigma e \sigma^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for a suitable permutation σ

from \mathcal{S}_3 . In fact, here, σ is the product of transpositions $(2, 3)(1, 3)$. We will prove this observation by explicitly constructing σ in an algorithmic manner.

If $i = 1$, then skip the next three sentences. Since e is an idempotent, the i -th entry in the i -th row of e has to be 1. Apply $\sigma_1 := (1, i)$ to bring the nonzero entry at the (i, i) -th position to $(1, 1)$ -th position. In fact, the application of σ_1 to e interchanges the i -th row

The proof of our next observation is now straightforward; it follows from the similar arguments as above.

Corollary 6.1. *Let e be an idempotent from \mathcal{P}_n . Then there exists a permutation $\sigma \in \mathcal{S}_n$ such that $\sigma e \sigma^{-1}$ is a block diagonal matrix of the form*

$$\sigma e \sigma^{-1} = c^{(\beta_1)} \oplus \cdots \oplus c^{(\beta_r)} \oplus \mathbf{0}_m, \quad (23)$$

where $\mathbf{0}_m$ is the square 0 matrix of dimension $m := n - \sum \beta_i$ and $\beta_1 \geq \beta_2 \cdots \geq \beta_r$.

We now generalize our previous definition of the “standard form of an idempotent” to its best: For an arbitrary idempotent $e \in \mathcal{P}_n$, we call its conjugate $\sigma e \sigma^{-1} = c^{(\beta_1)} \oplus \cdots \oplus c^{(\beta_r)} \oplus \mathbf{0}_m$ as in (23) the standard form of e and denote it by $st(e)$. An idempotent e is called *standard* if it satisfies the equation $x = st(x)$. Our next task is to compute the stabilizer of a standard idempotent.

Theorem 6.1. *Let $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_r$ be a sequence of positive integers and let $e \in \mathcal{P}_n$ be a standard idempotent in which $c^{(\beta_1)}$ appears m_1 times, $c^{(\beta_2)}$ appears m_2 times, and so on. If $\sum_{i=1}^s m_i \beta_i = n$, then the stabilizer subgroup in \mathcal{S}_n of e is*

$$\text{Stab}_{\mathcal{S}_n}(e) = (\mathcal{S}_{m_1} \wr \mathcal{S}_{\beta_1-1}) \times (\mathcal{S}_{m_2} \wr \mathcal{S}_{\beta_2-1}) \times \cdots \times (\mathcal{S}_{m_s} \wr \mathcal{S}_{\beta_s-1}).$$

Proof. We start with the simple case that $e = c^{(\beta_1)} \oplus \cdots \oplus c^{(\beta_1)} \oplus c^{(\beta_2)} \oplus \cdots \oplus c^{(\beta_2)}$, where $c^{(\beta_1)}$ appears m_1 times and $c^{(\beta_2)}$ appears m_2 times. If $\sigma \in \mathcal{S}_n$ stabilizes e , then we have $\sigma e = e \sigma$. We decompose σ into 4 blocks

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

where σ_{11} is of dimension $m_1 \beta_1$ and σ_{22} is of dimension $m_2 \beta_2$. For simplicity of notation, let us denote $c^{(\beta_1)} \oplus \cdots \oplus c^{(\beta_1)}$ by c_{11} and denote $c^{(\beta_2)} \oplus \cdots \oplus c^{(\beta_2)}$ by c_{22} . By a simple block-matrix multiplication we observe that

1. σ_{11} commutes with c_{11} ;
2. σ_{22} commutes with c_{22} ;
3. $\sigma_{12} c_{22} = c_{11} \sigma_{12}$ which implies that σ_{12} is a 0–matrix since $c^{(\beta_1)}$ and $c^{(\beta_2)}$ are of different dimensions;
4. $\sigma_{21} c_{11} = c_{22} \sigma_{21}$ which implies that σ_{21} is a 0–matrix since $c^{(\beta_1)}$ and $c^{(\beta_2)}$ are of different dimensions.

The last two items shows that $\text{Stab}_{\mathcal{S}_n}(e) = \text{Stab}_{\mathcal{S}_{m_1 \beta_1}}(c_{11}) \oplus \text{Stab}_{\mathcal{S}_{m_2 \beta_2}}(c_{22})$. Thus, σ_{ii} belongs to $\mathcal{S}_{m_i \beta_i}$ for $i = 1, 2$.

To show that $\text{Stab}_{\mathcal{S}_{m_i\beta_i}}(c_{ii}) \simeq \mathcal{S}_{m_i} \wr \mathcal{S}_{\beta_i-1}$ ($i = 1, 2$), we assume without loss of generality that $i = 1$ and we decompose σ_{11} into m_1^2 square blocks as in

$$\sigma_{11} = \begin{pmatrix} \tau_{11} & \cdots & \tau_{1m_1} \\ \vdots & \ddots & \vdots \\ \tau_{m_11} & \cdots & \tau_{m_1m_1} \end{pmatrix}. \quad (24)$$

Note that each τ_{ij} is a rook matrix. Since c_{11} has copies of $c^{(\beta_1)}$'s along its main diagonal and zero blocks elsewhere, it follows from the commutation relation $\sigma_{11}c_{11} = c_{11}\sigma_{11}$ that

$$\tau_{ij}c^{(\beta_1)} = c^{(\beta_1)}\tau_{ij} \quad (25)$$

for all $i, j = 1, \dots, m_1$. It is not difficult to verify that unless τ_{ij} is a permutation matrix with 1 at its $(1, 1)$ -th entry, in order for relation (25) to hold, τ_{ij} must be a zero matrix. In other words, in each row (of blocks) in (24) there exists exactly 1 block which is a permutation matrix of dimension β_1 with 1 at its $(1, 1)$ -th entry. It is not difficult to verify that a permutation matrix $\tau \in \mathcal{S}_{m_1\beta_1}$ with m_1^2 square blocks just as described satisfies the commutation relation $\tau c_{11} = c_{11}\tau$, also. But such a permutation τ is uniquely determined by a permutation $\tau' \in \mathcal{S}_{m_1}$ describing the positions of nonzero blocks of τ and in addition m_1^2 permutation matrices $\tau'_{ij} \in \mathcal{S}_{\beta_1-1}$ which forms that lower-right sub-matrices of the nonzero blocks. This is precisely our definition of $\mathcal{S}_{m_1} \wr \mathcal{S}_{\beta_1-1}$. \square

Corollary 6.2. *If $e \in \mathcal{P}_n$ is an idempotent, then there exists a multiset of integers $\{\beta_1^{m_1}, \dots, \beta_s^{m_s}\}$ and $m \in \mathbb{N}$ such that*

1. $m + \sum_{i=1}^s m_i\beta_i = n$;
2. $\text{Stab}_{\mathcal{S}_n}(e) \simeq (\mathcal{S}_{m_1} \wr \mathcal{S}_{\beta_1}) \times (\mathcal{S}_{m_2} \wr \mathcal{S}_{\beta_2}) \times \cdots \times (\mathcal{S}_{m_s} \wr \mathcal{S}_{\beta_s}) \times \mathcal{S}_m$.

Proof. We know from Corollary 6.1 that by conjugating with an appropriate permutation $\sigma \in \mathcal{S}_n$ we get $\sigma e \sigma^{-1} = \tilde{e} \oplus \mathbf{0}_m$, where \tilde{e} is a standard idempotent in \mathcal{P}_k for some $1 \leq k \leq n$ and $m = n - k$. The rest of the proof follows from Theorem 6.1. \square

7 The conjecture

Throughout this section, when the parts of a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ satisfy $\lambda_1 = \cdots = \lambda_k$, we will write $\lambda = \lambda_1^k$.

Let τ be a rooted forest with the Frobenius character F_τ . Recall from Introduction that adding a root to τ has the effect of multiplying F_τ with h_1 . Recall also that, if τ is a rooted tree, then the Frobenius character of the rooted forest that is comprised of m copies of τ is equal to $h_m[F_\tau]$.

Example 7.1. *Since h_1 is equal to the power sum symmetric function p_1 , the Frobenius character of $n - 1$ disjoint nodes (for $n \geq 2$) is $h_{n-1}[h_1] = h_{n-1}$.*

Example 7.2. *The Frobenius character of the rooted tree that is obtained from $n-1$ disjoint nodes (for $n \geq 2$) by attaching them to a root is $h_1 h_{n-1}$.*

Example 7.3. *The Frobenius character of the rooted forest that is obtained by taking m copies of the rooted tree of Example 7.2 is $h_m[h_1 h_{n-1}]$.*

The rest of our paper is devoted to a proof of the conjecture that is mentioned in the introduction. In the light of Examples 7.1–7.3, the conjecture translates to the symmetric function language as follows:

Conjecture 7.1. *if $n > m \geq 2$, then for any λ which has more than two parts,*

$$\langle h_n[h_1 h_{m-1}], s_\lambda \rangle \geq \langle h_m[h_1 h_{n-1}], s_\lambda \rangle.$$

We will prove this conjecture in the case where $m = 2$. In particular, we shall show that for all $n > 2$, and $\lambda \neq (2, 2n-2)$,

$$\langle h_n[h_1 h_1], s_\lambda \rangle \geq \langle h_2[h_1 h_{n-1}], s_\lambda \rangle. \quad (26)$$

and

$$\langle h_n[h_1 h_1], s_{(2, 2n-2)} \rangle = 2, \text{ and } \langle h_2[h_1 h_{n-1}], s_{(2, 2n-2)} \rangle = 3.$$

Before we can carry out our computations, we need to recall Littlewood’s formulas for the plethysms $h_n[h_2]$ and $h_n[e_2]$, the Murnaghan-Nakayama rule to compute the product of power symmetric function p_n times a Schur function s_μ , and Chen’s rule which allows us to compute the plethysm $p_k[h_n]$.

We say that a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ is even if λ_i is even for all i . Then Littlewood [5] proved the following two results.

$$h_n[h_2] = \sum_{\lambda \vdash 2n, \lambda \text{ even}} s_\lambda \quad (27)$$

and

$$h_n[e_2] = \sum_{\lambda \vdash 2n, \lambda' \text{ even}} s_\lambda \quad (28)$$

Next we introduce the notion of rim hook tableaux, special rim hook tabloids, and transposed special rim hook tabloids.

Given a Ferrers diagram $F(\lambda)$ of a partition, a *rim hook* h of λ is a consecutive sequence of cells along the northeast boundary of λ such that any two consecutive cells of h share an edge and the removal of the cells of h from λ results in a Ferrers diagram of another partition. We let $r(h)$ denote the number of rows of h and $c(h)$ denote the number of columns of h . We say that h is *special* if h has at least one cell in the first column of $F(\lambda)$ and h is *transposed special* if h has at least one cell in the first row of $F(\lambda)$. We define the sign of h , $sgn(h)$, to be

$$sgn(h) = (-1)^{r(h)-1}.$$

T , namely,

$$\begin{aligned} T &= \{\emptyset \subset (1, 2) \subset (2, 2, 2) \subset (2, 2, 4)\} \text{ and} \\ T &= \{\emptyset \subset (1, 2) \subset (1, 4) \subset (2, 2, 4)\} \end{aligned}$$

depending on whether one takes to top rim hook to be the second or third rim hook. Of course, if T is a special rim hook tabloid or a t -special rim hook tabloid, then there is a unique sequence of shapes $\{\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \dots \subset \lambda^{(k)} = \lambda\}$ that can be associated to T .

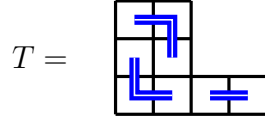


Figure 9: A rim hook tabloid with associated sequences of shapes.

Chen's rule [2] states that

$$\langle p_k[h_n], s_\lambda \rangle = \begin{cases} \text{sgn}(T) & \text{if } T \text{ is a special rim hook tabloid of shape } \lambda \text{ and type } k^n, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \quad (30)$$

Thus to compute $p_k[s_n] = s_n[p_k]$, we need only generate all the t -special rim hook tabloids T of type (k^n) and replace T by $\text{sgn}(T)s_{sh(T)}$ where $sh(T)$ denotes the shape of T . For example, Figure 10 pictures all the t -special rim hook tabloids of type 3^2 where instead of drawing a Ferrers diagram, we have indicated the cells of the Ferrers diagram by dots. Thus $p_3[h_2] = s_2[p_3] = s_{(2^3)} - s_{(1,2,3)} + s_{(1^2,4)} + s_{(3,3)} - s_{(1,5)} + s_{(6)}$.

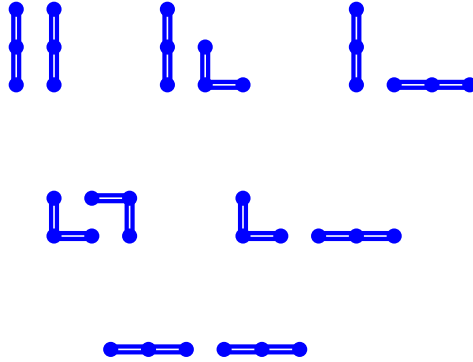


Figure 10: The t -special rim hook tabloids of type (3^2) .

The computation of $p_2[h_n]$ is particular straightforward via Chen's rule. That is, a t -special rim hook tabloid of type 2^n consists of bunch of vertical rim hooks followed by a bunch of horizontal rim hooks as pictured in Figure 11.

It easily follows that

$$p_2[s_n] = \sum_{0 \leq a \leq n} (-1)^a s_{(a, 2n-a)}. \quad (31)$$

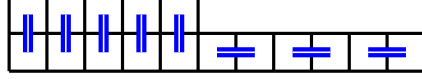


Figure 11: t -special rim hook tabloids of type 2^n .

Although we are specifically interested in understanding the role of multiplication by a complete symmetric function then applying plethysm, in our computations we will often write p_1 in place of h_1 . Also, the following notation will be handy: For any logical statement A , we set

$$\chi(A) = \begin{cases} 1 & \text{if } A \text{ is true;} \\ 0 & \text{if } A \text{ is false.} \end{cases}$$

We want to compare $h_2[p_1 h_{n-1}]$ to $h_n[p_1 h_1]$. Obviously, the two expressions are equal when $n = 2$. Our first goal is to prove the following theorem which gives an explicit formula for the Schur function expansion of $h_2[p_1 h_{n-1}]$.

Theorem 7.1. *For $n \geq 3$,*

$$\begin{aligned} h_2[h_1 h_{n-1}] &= \sum_{1 \leq a \leq n-1, a \text{ odd}} s_{(1^2, a, 2n-2-a)} + \sum_{2 \leq a \leq n-1, a \text{ even}} s_{(2, a, 2n-2-a)} + \\ & s_{(1^2, 2n-2)} + \sum_{2 \leq a \leq n-1} 2s_{(1, a, 2n-1-a)} + \\ & s_{(1, 2n-1)} + \sum_{2 \leq a \leq n-1, a \text{ even}} 3s_{(a, 2n-a)} + \sum_{2 \leq a \leq n-1, a \text{ odd}} s_{(a, 2n-a)} + \\ & 2\chi(n \text{ even})s_{(n, n)} + s_{(2n)}. \end{aligned}$$

Proof. Note that by (31) and the Pieri rules we have

$$\begin{aligned} h_2[p_1 h_{n-1}] &= \left(\frac{1}{2}(p_1^2 + p_2) \right) [p_1 h_{n-1}] \\ &= \frac{1}{2}p_1^2[p_1 h_{n-1}] + \frac{1}{2}p_2[p_1 h_{n-1}] \\ &= \frac{1}{2}p_1^2 h_{n-1}^2 + \frac{1}{2}p_2 p_2 [h_{n-1}] \\ &= \frac{1}{2} \left(p_1^2 \left(\sum_{0 \leq a \leq n-1} s_{(a, 2n-2-a)} \right) + p_2 \left(\sum_{0 \leq a \leq n-1} (-1)^a s_{(a, 2n-2-a)} \right) \right). \end{aligned}$$

It is easy to see from the Pieri rules and the Murnaghan-Nakayama rules that the only shapes λ of Schur functions s_λ that can arise in the Schur function expansions of products of the form $p_1^2 s_{(a, 2n-2-a)}$ and $p_2 s_{(a, 2n-2-a)}$ are $(1^2, a, b)$, $(2, a, b)$, $(1, a, b)$, (a, b) , or $(2n)$. We will consider each type of shape in turn. It follows from the Pieri rules that a product $p_1^2 s_{(a, 2n-2-a)}$ is the sum over all s_λ where we first add single cell on the outside of $(a, 2n-2-a)$ to get a

shape μ , which we will indicate by a cell containing the number 1, and then adding a single cell on the outside of the μ , which we will indicate by cell containing 2.

By the Murnaghan-Nakayama rule, a product of the form $p_2 s_{(a, 2n-2-a)}$ is the sum over all $\pm s_\lambda$ that arises by putting a rim hook of length 2 along the outside of $(a, 2n-2-a)$, which we will indicate by a rim hook colored red, where the sign is plus if the rim hook is horizontal and -1 if the rim hook is vertical.

Case 1. $\lambda = (1^2, a, b)$ where $1 \leq a \leq n-1$

In this case, there is only one way to get a term of the form where $1 \leq a \leq n-1$ from the products $p_1^2 h_{n-1}^2$ and $p_2[h_{n-1}]p_2$ which are pictured in Figure 12. That is, it is easy to see that $s_{(1^2, a, 2n-2-a)}$ can only arise in one way from $p_1^2 h_{n-1}^2$ which comes the product $p_1^2 s_{(a, 2n-2-a)}$ by placing the cells with 1 and 2 as pictured at the top of Figure 12. Similarly, $s_{(1^2, a, 2n-2-a)}$ can only arise in one way from which comes the product $p_2(-1)^a s_{(a, 2n-2-a)}$ by placing the rim hook as pictured in red at the bottom of Figure 12. Hence we get a contribution of $(-1)^{a+1} s_{(1^2, a, 2n-2-a)}$ from $p_2 p_2[h_{n-1}]$. These two cancel out if $1 \leq a \leq n-1$ and a is even and they give a contribution of $\frac{1}{2} 2 s_{(1^2, a, 2n-2-a)} = s_{(1^2, a, 2n-2-a)}$ if a is odd.

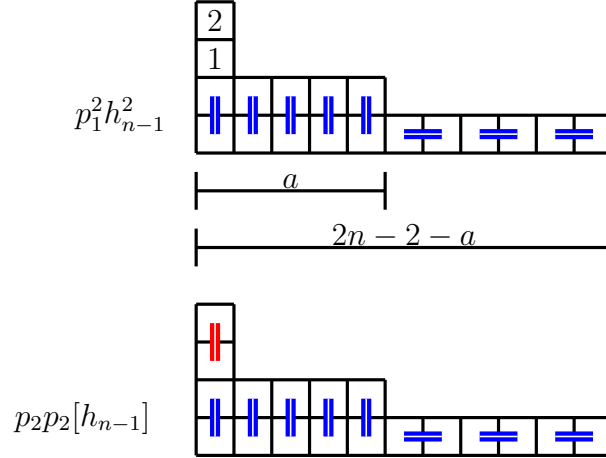


Figure 12: The diagrams for Case 1.

Thus the Schur functions in Case 1 contribute

$$\sum_{1 \leq a \leq n-1, a \text{ odd}} s_{(1^2, a, 2n-2-a)}$$

to $h_2[h_1 h_{n-1}]$.

Case 2. $\lambda = (2, a, b)$ where $2 \leq a \leq n-1$

In this case, there is only one way to get a term of the form where $1 \leq a \leq n-1$ from the products $p_1^2 h_{n-1}^2$ and $p_2[h_{n-1}]p_2$ which are pictured in Figure 13. That is, it is easy to see that

$s_{(1^2, a, 2n-2-a)}$ can only arise in one way from $p_1^2 h_{n-1}^2$ which comes the product $p_1^2 s_{(a, 2n-2-a)}$ by placing the cells with 1 and 2 as pictured at the top of Figure 13. Similarly, $s_{(1^2, a, 2n-2-a)}$ can only arise in one way from which comes the product $p_2(-1)^a s_{(a, 2n-2-a)}$ by placing the rim hook as pictured in red at the bottom of Figure 13. Hence we get a contribution of $(-1)^a s_{(1^2, a, 2n-2-a)}$ from $p_2 p_2[h_{n-1}]$. This two cancel out if $1 \leq a \leq n-1$ and a is odd and a contribution of $\frac{1}{2} 2 s_{(1^2, a, 2n-2-a)} = s_{(1^2, a, 2n-2-a)}$ if a is even.

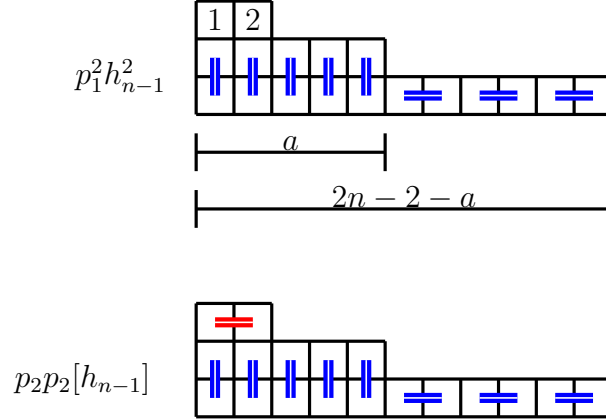


Figure 13: The diagrams for Case 2.

Thus the Schur functions in Case 2 contribute

$$\sum_{2 \leq a \leq n-1, a \text{ even}} s_{(2, a, 2n-2-a)}$$

to $h_2[p_1 h_{n-1}]$.

Case 3. $\lambda = (1, a, 2n-a-1)$ where $1 \leq a \leq n-1$.

In this case, we must consider two subcases.

Subcase 3.1 $a = 1$.

Then $s_{(1^2, 2n-2-a)}$ can arise in exactly three ways from $p_1^2 h_{n-1}^2$, once from $p_1^2 s_{2n-2}$ and two ways from $p_1^2 s_{(1, 2n-3)}$. These are pictured at the top of Figure 14. $s_{(1^2, 2n-2-a)}$ can only arise in one way from $p_2 p_2[h_{n-1}]$ which from the product $p_2 s_{2n-2}$ and comes with a sign of -1 . This possibility is pictured at the bottom Figure 14. Thus $s_{(1^2, 2n-2)}$ appears once in the expansion of $h_2[p_1 h_{n-1}]$.

Subcase 3.2. $2 \leq a \leq n-1$.

Then it is easy to see that we can not obtain a Schur function of shape $(1, a, 2n-1-a)$ by adding a rim hook of size two on the outside of two row shape. Thus $s_{(1, a, 2n-1-a)}$ does not

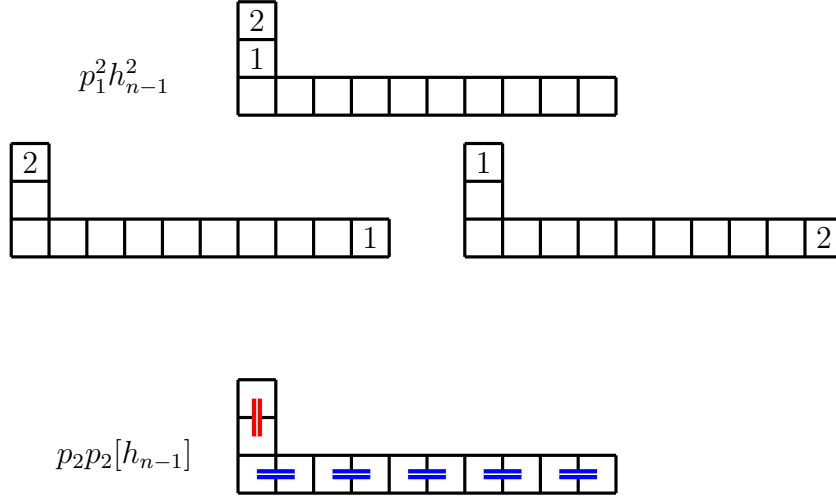


Figure 14: The diagrams for Case 3.1.

appear in the expansion of $p_2[h_{n-1}]p_2$. However it can appear in the expansion of $p_1^2 h_{n-1}$ in 4 different ways, namely, twice from the product $p_1^2 s_{(a-1, 2n-a-1)}$ and twice from the product $p_1^2 s_{(a, 2n-a-2)}$. These are pictured in Figure 15. Thus in Subcase 3.2, we get an additional contribution of $2 \sum_{2 \leq a \leq n-1} s_{(1, a, 2n-1-a)}$.

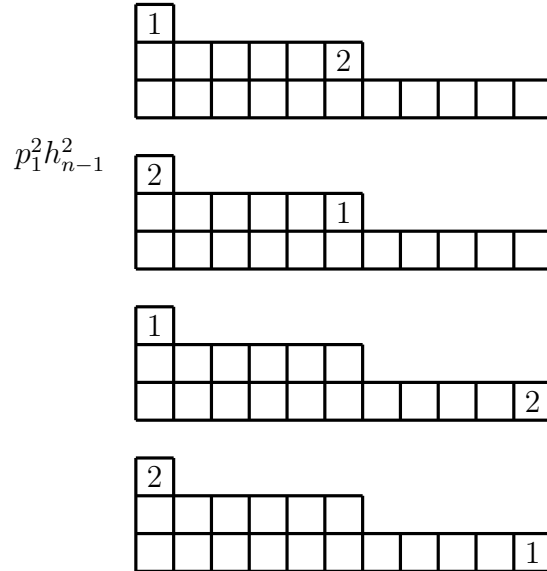


Figure 15: The diagrams for Case 3.2.

Thus the Schur functions in Case 3 contribute

$$s_{(1^2, 2n-2)} + 2 \sum_{2 \leq a \leq n-1} s_{(1, a, 2n-1-a)}$$

to $h_2[p_1 h_{n-1}]$.

Case 4. $\lambda = (a, 2n - a)$ where $1 \leq a \leq n$.

In this case, we have three subcases.

Subcase 4.1 $a = 1$.

In this case we have three ways to obtain $s_{(1,2n-1)}$ from $p_1^2 h_{n-1}^2$, namely, two ways from the product $p_1^2 s_{(2n-1)}$ and one way from the product $p_1^2 s_{(1,2n-3)}$. These three possibilities are pictured at the top of Figure 16. There is only one way to obtain $s_{(1,2n-1)}$ from $p_2 p_2[h_{n-1}]$, namely, from the product $p_2 s_{(1,2n-3)}$ which comes with a sign of -1 . Thus Subcase 4.1 contributes a factor of $s_{(1,2n-1)}$.

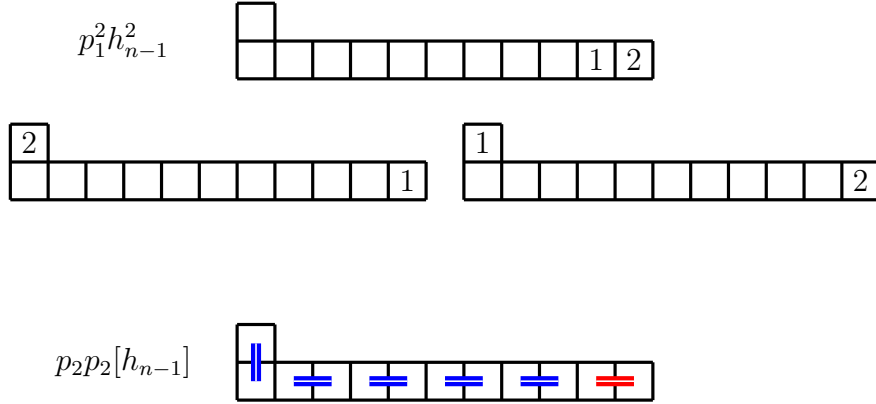


Figure 16: The diagrams for Case 4.1.

Subcase 4.2 $a = n$.

In this case we have two ways to obtain $s_{(n,n)}$ from $p_1^2 h_{n-1}^2$, namely, one the product $p_1^2 s_{(n-2,n)}$ and one way from the product $p_1^2 s_{(n-1,n-1)}$. These possibilities are pictured at the top of Figure 17. There are two ways to obtain $s_{(n,n)}$ from $p_2 p_2[h_{n-1}]$, namely, once from the product $p_2 s_{(n-2,n)}$ which comes with a sign of $(-1)^{n-2}$ and once from the product $p_2 s_{(n-1,n-1)}$ which comes with a sign of $(-1)^n$. These possibilities are pictured at the bottom of Figure 17. Thus Subcase 4.2 contributes a factor of $2s_{(n,n)}$ if n is even and 0 if n is odd.

Subcase 4.3 $2 \leq a \leq n - 1$.

In this case we have four ways to obtain $s_{(a,2n-a)}$ from $p_1^2 h_{n-1}^2$, namely, once from the product $p_1^2 s_{(a-2,2n-a)}$, once from the product $p_1^2 s_{(a,2n-a-2)}$, and twice from the product $p_1^2 s_{(a-1,2n-a-1)}$. These possibilities are pictured at the top of Figure 18. There are two ways to obtain $s_{(a,2n-1)}$ from $p_2 p_2[h_{n-1}]$, namely, once from the product $p_2 s_{(a-2,2n-a)}$ which comes with a sign of $(-1)^{a-2}$ and once from the product $p_2 s_{(a,2n-a-2)}$ which comes with a sign of $(-1)^a$. These

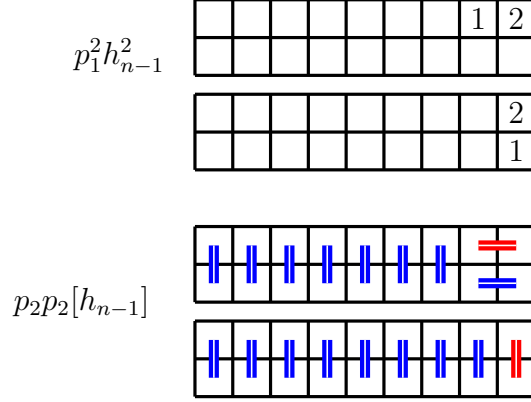


Figure 17: The diagrams for Case 4.2.

possibilities are pictured at the bottom of Figure 18. Thus Subcase 4.3 contributes a factor of

$$\sum_{2 \leq a \leq n-1, a \text{ even}} 3s_{(a, 2n-a)} + \sum_{2 \leq a \leq n-1, a \text{ odd}} s_{(a, 2n-a)}.$$

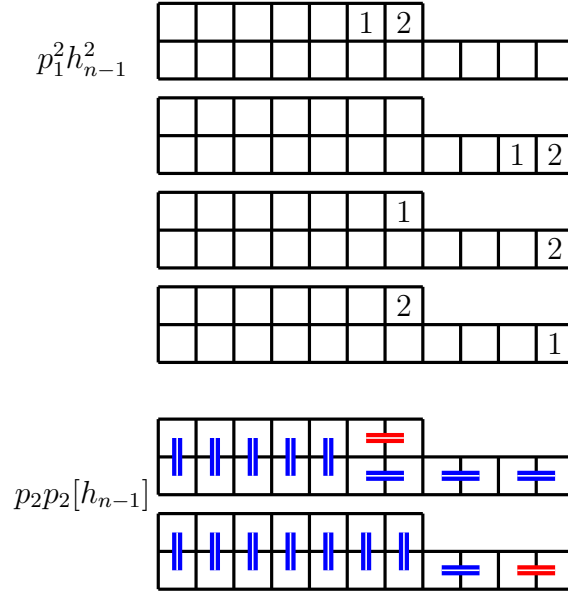


Figure 18: The diagrams for Case 4.3.

Thus the Schur functions in Case 4 contribute

$$2\chi(n \text{ is even})s_{(n,n)} + 3 \sum_{2 \leq a \leq n-1, a \text{ even}} s_{(a, 2n-a)} + \sum_{1 \leq a \leq n-1, a \text{ odd}} s_{(a, 2n-a)}$$

to $h_2[p_1 h_{n-1}]$.

Case 5. $\lambda = (2n)$.

In this case, we have one way to obtain $s_{(2n)}$ from $p_1^2 h_{n-1}^2$, one way from the product $p_1^2 s_{(2n-2)}$. There also is only one way to obtain $s_{(2n)}$ from $p_2 p_2 [h_{n-1}]$, namely, from the product $p_2 s_{(2n-2)}$ which comes with a sign of 1. Thus Subcase 5 contributes a factor of $s_{(2n)}$ to $h_2 [p_1 h_{n-1}]$. \square

Note that

$$\begin{aligned} h_n [p_1 h_1] &= h_n [h_2 + e_2] \\ &= \sum_{k=0}^n h_k [h_2] h_{n-k} [e_2] \\ &= \sum_{k=0}^n \left(\sum_{\lambda \text{ even}, \lambda \vdash 2k} s_\lambda \right) \left(\sum_{\lambda' \text{ even}, \lambda' \vdash 2(n-k)} s_{\lambda'} \right). \end{aligned}$$

Our goal is to show that for all partitions of $2n$ except $\lambda = (2, 2n-2)$,

$$\langle h_n [h_1 h_1], s_\lambda \rangle \geq \langle h_2 [h_1 h_{n-1}], s_\lambda \rangle$$

It follows from our previous theorem that the only s_λ such that $\langle h_2 [h_1 h_{n-1}], s_\lambda \rangle \neq 0$ is when λ is either contained in $(2, 2n, 2n)$ or $(1^2, 2n, 2n)$. Now the only such s_λ that appear in $\left(\sum_{\lambda \text{ even}, \lambda \vdash 2k} s_\lambda \right)$ are of the form

1. $s_{(2, a, 2k-a-2)}$ where a is even and $2 \leq a \leq n-1$,
2. $s_{(a, 2k-a)}$ where a is even and $2 \leq a \leq n$, or
3. $s_{(2k)}$.

The only such s_λ that appear in $\left(\sum_{\lambda' \text{ even}, \lambda' \vdash 2(n-k)} s_{\lambda'} \right)$ are of the form

1. $s_{(1^2, n-k-1, n-k-1)}$ or
2. $s_{(n-k, n-k)}$.

Our goal now is to go through the six cases of the proof of Theorem 7.1 to show that in all but one case, $\langle h_n [p_1 h_1], s_\lambda \rangle \geq \langle h_2 [p_1 h_{n-1}], s_\lambda \rangle$. However, we will need to apply the skew Schur function expansion rule of Remmel and Whitney [7]. Given a skew shape, λ/μ written in French notation, the reverse lexicographic filling of λ/μ , $RLF(\lambda/\mu)$, is the filling of the Ferrers diagram with the integers $1, \dots, |\lambda/\mu|$, which is obtained by filling the cells in order from right to left in each row, starting at the bottom row and proceeding upwards. For example, the $RLF((2, 4, 4, 4)/(1, 2))$ is pictured in Figure 19.

A standard tableau T of shape ν is said to λ/μ -compatible if and only if T satisfies the following two conditions.

11	10		
9	8	7	6
	5	4	3
		2	1

Figure 19: The reverse lexicographic filling of $(2, 4, 4, 4)/(1, 2)$.

- R1.** If x and $x + 1$ lie in the same row in $RLF(\lambda/\mu)$, then $x + 1$ appear strictly to the right and weakly below x in T .
- R2.** If y is immediately above x in $RLF(\lambda/\mu)$, then y must appear strictly above and weakly to the left of x in T .

A pictorial representation of these two conditions are give in Figure 20. We let $\lambda/\mu\text{-ST}$ denote the set of all λ/μ -compatible standard tableaux.

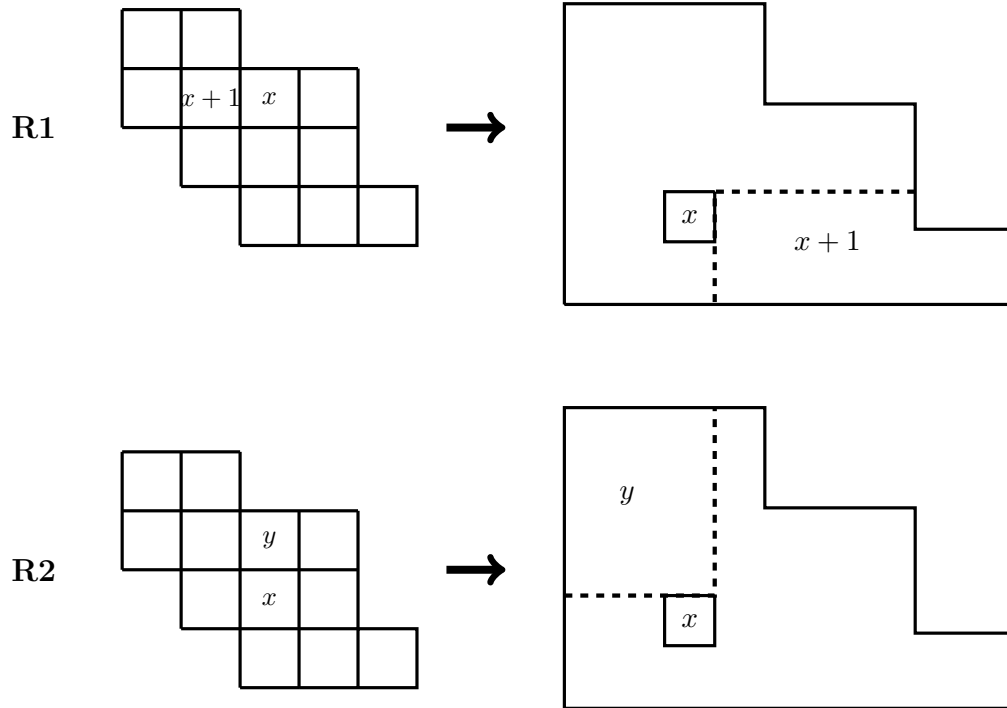


Figure 20: The two rules for λ/μ -compatible standard tableaux.

The the **skew Schur function expansion rule** is that

$$s_{\lambda/\mu} = \sum_{T \in \lambda/\mu\text{-ST}} s_{sh(T)}$$

where $sh(T)$ is the shape of T . For example, if $\lambda = (2, 2, 3, 4)$ and $\mu = (1, 2, 2)$, then in Figure 21, we have pictured the $RLF(\lambda/\mu)$ at the top right and we have constructed a tree

which generates all T in $\lambda/\mu\text{-ST}$ by adding a new element at each stage to obey rules R1 and R2. It then follows in this case that

$$s_{(2,2,3,4)/(1,2,2)} = s_{(1,1,2,2)} + s_{(1,1,1,3)} + s_{(2,2,2)} + 2s_{(1,2,3)} + s_{(1,1,4)} + s_{(3,3)} + s_{(2,4)}.$$

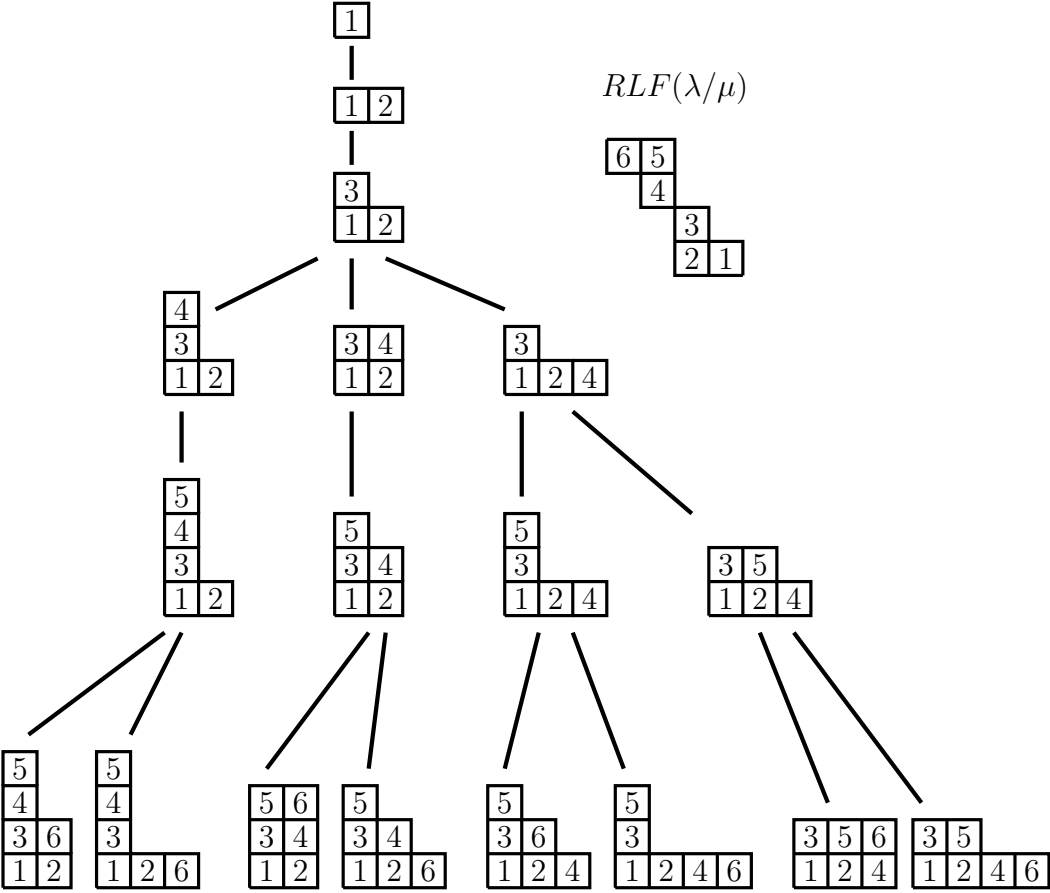


Figure 21: The tree for the expansion of $s_{(2,2,3,4)/(1,2,2)}$.

Note that expanding a product of Schur functions is a special case of expanding a skew Schur function. That is, it is easy to see that $s_\lambda s_\mu$ is equal to the skew Schur function $s_{\lambda * \mu}$ where $\lambda * \mu$ is the skew shape that arises by placing λ above μ so that the bottom row of λ starts just to the left of the top row of μ . For example, if $\lambda = (2, 2)$ and $\mu = (1, 2)$, then $\lambda * \mu$ is pictured at the top-right of Figure 22. The other thing to observe is that if $\mu = (\mu_1, \dots, \mu_k)$ where $0 < \mu_1 \leq \dots \leq \mu_k$, then in the expansion of $s_{\lambda * \mu}$, the numbers in the part of the reverse lexicographic filling of μ are completely determined. That is, $1, \dots, \mu_k$ must appear in row 1, reading from left to right, the numbers $\mu_k + 1, \dots, \mu_k + \mu_{k-1}$ must appear in row 2, reading from left to right, the numbers $\mu_k + \mu_{k-1} + 1, \dots, \mu_k + \mu_{k-1} + \mu_{k-2}$ must appear in row three, etc.. This means that in constructing our tree, we will eventually

reach a single filling of shape μ and then we will continue to build our tree using the numbers of the reverse lexicographic filling that corresponding to λ . One can see an example of this phenomenon in Figure 21 in processing the numbers 1, 2, and 3 corresponding to the shape $(1, 2)$. This means that we might as well ignore the numbers in μ and simply build our tree starting with a the Ferrers diagram of μ and using the numbers $1, 2, \dots, |\lambda|$ to fill the part corresponding to λ in reverse lexicographic filling of $\lambda * \mu$. An example of this process is pictured in Figure 22 for the Schur function expansion of $s_{(2,2)}s_{(1,2)}$. In this case, we see that

$$s_{(2,2)}s_{(1,2)} = s_{(1,2,2,2)} + s_{(1,1,2,3)} + s_{(2,2,3)} + s_{(1,3,3)} + s_{(1,2,4)} + s_{(3,4)}.$$

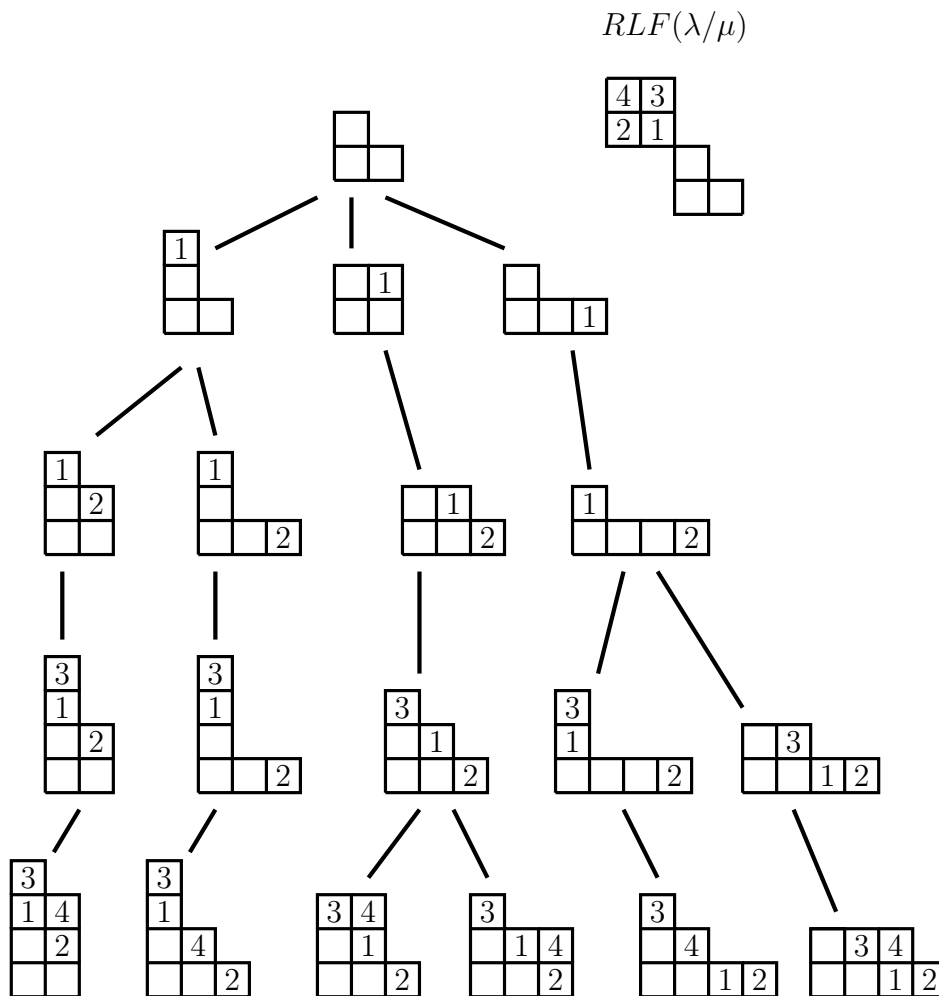


Figure 22: The tree for the expansion of $s_{(2,2)}s_{(1,2)}$.

Case 1. $\lambda = (1^2, a, 2n - 2 - a)$ where a is odd and $1 \leq a \leq n - 1$.

In this case, we have shown that $\langle h_2[p_1 h_{n-1}], s_{(1^2, a, 2n-a-2)} \rangle = 1$. It is easy to see from the Pieri rule that $s_{(1^2, a, 2n-a-2)}$ appears in the expansion of $s_{1^4} s_{(a-1, 2n-2-a-1)}$ with coefficient 1. But the term $s_{(1^4)}$ appears in the expansion of $h_2[e_2]$ and the term $s_{(a-1, 2n-2-a-1)}$ appears in the expansion of $h_{n-2}[h_2]$ when a is odd. Thus when a is odd,

$$\langle h_n[p_1 h_1], s_{(1^2, a, 2n-a-2)} \rangle \geq 1 \geq \langle h_2[p_1 h_{n-1}], s_{(1^2, a, 2n-a-2)} \rangle.$$

Case 2. $\lambda = (2, a, 2n - 2 - a)$ where a is even and $2 \leq a \leq n - 1$.

In this case, we have shown that $\langle h_2[p_1 h_{n-1}], s_{(2, a, 2n-a-2)} \rangle = 1$. Since $(2, a, 2n - a - 2)$ has even parts, $s_{(2, a, 2n-a-2)}$ appears in the expansion of $h_n[h_2]$ so that

$$\langle h_n[p_1 h_1], s_{(2, a, 2n-a-2)} \rangle \geq 1 \geq \langle h_2[p_1 h_{n-1}], s_{(2, a, 2n-a-2)} \rangle.$$

Case 3. $\lambda = (1, a, 2n - 1 - a)$ where $1 \leq a \leq n - 1$.

First we consider the case where $a = 1$. In this case, we have shown that $\langle h_2[p_1 h_{n-1}], s_{(1^2, 2n-2)} \rangle = 1$. It is easy to see from the Pieri rule that $s_{(1^2, 2n-2)}$ appears in the expansion of $s_{1^2} s_{(2n-2)}$ with coefficient 1. But the term s_{1^2} appears in the expansion of $h_1[e_2]$ and the term $s_{(2n-2)}$ appears in the expansion of $h_{n-1}[h_2]$. Thus

$$\langle h_n[p_1 h_1], s_{(1^2, 2n-2)} \rangle \geq 1 \geq \langle h_2[p_1 h_{n-1}], s_{(1^2, 2n-2)} \rangle.$$

When $2 \leq a \leq n - 1$, we have shown that $\langle h_2[p_1 h_{n-1}], s_{(1, a, 2n-1-a)} \rangle = 2$. The first thing to observe is that when a is even, then $s_{(a, 2n-2-a)}$ appears in the expansion of $h_{n-1}[h_2]$ from which it follows that $s_{(1, a, 2n-1-a)}$ will occur with multiplicity 1 in the Schur function expansion of $h_1[e_2]h_{n-1}[h_2] = s_{(1^2)}h_{n-1}[h_2]$. Similarly, when a is odd, then $s_{(a-1, 2n-1-a)}$ appears in the expansion of $h_{n-1}[h_2]$ from which it follows that $s_{(1, a, 2n-1-a)}$ will occur with multiplicity 1 in the Schur function expansion of $h_1[e_2]h_{n-1}[h_2] = s_{(1^2)}h_{n-1}[h_2]$. This situation is pictured at the top of Figure 23. If $2 \leq a \leq n - 1$, then $s_{(1, a, 2n-1-a)}$ will occur in the expansion of $s_{(2, 2)}h_{n-2}[h_2]$ as is evidenced by the pictures in the middle row of Figure 23. Note that in the case where $a = n - 1$ and a is odd, the expansion still works. It will just be the case that the 4 is on top of the 1. Hence in this case we get another occurrence of $s_{(1, a, 2n-1-a)}$ from the term $h_2[e_2]h_{n-2}[h_2]$. This will show that for $2 \leq a \leq n - 1$,

$$\langle h_n[p_1 h_1], s_{(1, a, 2n-1-a)} \rangle \geq 2 \geq \langle h_2[p_1 h_{n-1}], s_{(1^2, 2n-2)} \rangle.$$

In fact, in the case where $a = n - 1$, we note that $s_{(n-1, n-1)}$ appears in the expansion of $h_{n-1}[e_2]$ which implies that we would get another copy of $s_{(1, n-1, n)}$ in the expansion of $h_1[h_2]h_{n-1}[e_2] = h_2h_{n-1}[e_2]$ as pictured at the bottom of Figure 23.

It follows that if $\lambda = (a, 2n - a)$, then we have

$$\begin{aligned} \left\langle \sum_{k=0}^n h_k[h_2]h_{n-k}[e_2], s_{(a,2n-a)} \right\rangle &= \left\langle \sum_{k=0}^n h_k[h_2]s_{(n-k,n-k)}, s_{(a,2n-a)} \right\rangle \\ &= \sum_{k=0}^n \chi(n-k < a) \ \& \ (a - (n-k), 2n - a - (n-k)) \text{ is even.} \end{aligned}$$

Thus

$$\left\langle \sum_{k=0}^n h_k[h_2]h_{n-k}[e_2], s_{(a,2n-a)} \right\rangle = \begin{cases} \frac{a}{2} + 1 & \text{if } a \text{ is even} \\ \frac{a+1}{2} & \text{if } a \text{ is odd.} \end{cases} \quad (32)$$

It follows that if a is odd and $1 \leq a \leq n - 1$, then

$$\langle h_n[p_1 h_1], s_{(a,2n-a)} \rangle \geq 1 = \langle h_2[p_1 h_{n-1}], s_{(a,2n-a)} \rangle.$$

If a is even and $4 \leq a \leq n - 1$, then

$$\langle h_n[p_1 h_1], s_{(a,2n-a)} \rangle \geq 3 = \langle h_2[p_1 h_{n-1}], s_{(a,2n-a)} \rangle.$$

However, if $a = 2$, then $\langle h_n[p_1 h_1], s_{(a,2n-a)} \rangle = 2$ while $\langle h_2[p_1 h_{n-1}], s_{(a,2n-a)} \rangle = 3$ so we do not have

$$\langle h_n[p_1 h_1], s_{(2,2n-2)} \rangle \geq \langle h_2[p_1 h_{n-1}], s_{(2,2n-2)} \rangle.$$

The only other Schur function corresponding to a two row shape that we have not considered is $s_{(n,n)}$. In this case,

$$\langle h_2[p_1 h_{n-1}], s_{(n,n)} \rangle = \begin{cases} 0 & \text{if } a \text{ is odd} \\ 2 & \text{if } a \text{ is even.} \end{cases} \quad (33)$$

Note however, if n is even, then $s_{(n,n)}$ occurs in the expansion of $h_n[h_2]$ and the expansion of $h_n[e_2]$ so that

$$\langle h_n[p_1 h_1], s_{(n,n)} \rangle \geq 2 = \langle h_2[p_1 h_{n-1}], s_{(n,n)} \rangle.$$

Case 5. $\lambda = (2n)$.

In this case, we proved that $\langle h_2[p_1 h_{n-1}], s_{(2n)} \rangle = 1$. But $s_{(2n)}$ clearly appears in the expansion of $h_n[h_2]$ so that

$$\langle h_n[p_1 h_1], s_{(2n)} \rangle \geq \langle h_2[p_1 h_{n-1}], s_{(2n)} \rangle.$$

References

- [1] Mahir Bilen Can. A representation on labeled rooted forests. Submitted for publication, <https://arxiv.org/abs/1506.01543v2> (2017).
- [2] Y. M. Chen, A. M. Garsia, and J. Remmel. Algorithms for plethysm. In *Combinatorics and algebra (Boulder, Colo., 1983)*, volume 34 of *Contemp. Math.*, pages 109–153. Amer. Math. Soc., Providence, RI, 1984.
- [3] Louis Comtet. *Advanced combinatorics*. D. Reidel Publishing Co., Dordrecht, enlarged edition, 1974. The art of finite and infinite expansions.
- [4] Olexandr Ganyushkin and Volodymyr Mazorchuk. *Classical finite transformation semi-groups*, volume 9 of *Algebra and Applications*. Springer-Verlag London, Ltd., London, 2009. An introduction.
- [5] Dudley E. Littlewood. *The Theory of Group Characters and Matrix Representations of Groups*. Oxford University Press, New York, 1940.
- [6] Nicholas A. Loehr and Jeffrey B. Remmel. A computational and combinatorial exposé of plethystic calculus. *J. Algebraic Combin.*, 33(2):163–198, 2011.
- [7] J. B. Remmel and R. Whitney. Multiplying Schur functions. *J. Algorithms*, 5(4):471–487, 1984.