Abstract

This paper studies the combinatorics of the orbit Hecke algebras associated with $W \times W$ orbits in the Renner monoid of a finite monoid of Lie type, $M$, where $W$ is the Weyl group associated with $M$. It is shown by Putcha in [12] that the Kazhdan-Lusztig involution [6] can be extended to the orbit Hecke algebra which enables one to define the $R$-polynomials of the intervals contained in a given orbit. Using the $R$-polynomials, we calculate the Möbius function of the Bruhat-Chevalley ordering on the orbits. Furthermore, we provide a necessary condition for an interval contained in a given orbit to be isomorphic to an interval in some Weyl group.

1 Introduction

Let $G = G(F_q)$ be a finite group of Lie type (see [4]), $B \subseteq G$ a Borel subgroup, $T \subseteq B$ a maximal torus, and $W$ the Weyl group of the pair $(G, T)$. Set

$$\epsilon = \frac{1}{|B|} \sum_{g \in B} g.$$  \hspace{1cm} (1.1)
By a fundamental theorem of Tits, it is known that the algebra $\epsilon \mathbb{C}[G] \epsilon$ is isomorphic to the group algebra $\mathbb{C}[W]$.

The generic Hecke algebra $\mathcal{H}(W)$ of $G$, which is a deformation of $\mathbb{C}[W] \cong \epsilon \mathbb{C}[G] \epsilon$ is a fundamental tool in combinatorics, geometry and representation theory of $G$.

In [20], Solomon introduces the first example of a Hecke algebra for monoids in the special case of the monoid of $n \times n$ matrices over a finite field. In a series of papers ([10], [12], [13]), Putcha extends the theory of Hecke algebras of matrices to all finite regular monoids. In particular, he defines the orbit Hecke algebra $\mathcal{H}(J)$ for a $J$-class $J$ in a finite regular monoid.

Finite monoids of Lie type are regular monoids. Let $M$ be a finite monoid of Lie type and $G \subseteq M$ be its group of invertible elements. The group $G$ is a finite group of Lie type. Let $J$ be a $J$-class. Then, $J \subseteq M$ is a $G \times G$-orbit of the form $J = GeG$ for some idempotent $e$ of $M$. We write $\mathcal{H}(e)$ for $\mathcal{H}(J)$.

For the orbit $J = GeG$, the generic orbit Hecke algebra $\mathcal{H}(e)$ plays the role of the Hecke algebra $\mathcal{H}(W)$ for the group $G$. The algebra $\mathcal{H}(e)$ is a deformation of the contracted semigroup algebra $\epsilon \mathbb{C}_0[J^0] \epsilon$. Here, $\epsilon$ is as in (1.1), and $\mathbb{C}_0[J^0]$ is the semigroup algebra of the orbit semigroup $J^0 := J \cup \{0\}$ defined by

$$a \cdot b = \begin{cases} ab & \text{if } ab \in J, \\ 0 & \text{otherwise}. \end{cases}$$

As a $\mathbb{Z}[q^{-\frac{1}{2}}, q^{\frac{1}{2}}]$-algebra, $\mathcal{H}(W)$ is generated by a set of formal variables $\{T_w\}_{w \in W}$ indexed by the Weyl group $W$ and obeys a corresponding multiplication rule. In [12], Putcha extends the Kazhdan-Lusztig involution $\tau$ defined by $T_w = T_{w^{-1}}^{-1}$ for $w \in W$ to generic orbit Hecke algebras. Using this involution, he defines analogues of $R$-polynomials and $P$-polynomials of [6]. In this article, we investigate the combinatorial properties of Putcha’s $R$-polynomials. We show that given an interval in the Bruhat-Chevalley-Renner ordering, contained in an orbit $WeW$ inside the Renner monoid $R$ of the monoid $M$ ([16]), the constant term of the corresponding $R$-polynomial equals the value of the Möbius function on the given interval. Using this observation, we establish a criterion for when a subinterval of $WeW$ can be embedded into a Weyl group as a subinterval.
2 Background

The monoids of Lie type are introduced and classified by Putcha in [9] and [11]. Among the important examples of these monoids are the finite reductive monoids, [17]. For more information, interested reader may consult [8] and [18]. For an easy introduction to the theory reductive monoids we especially recommend the exposé by L. Solomon [19].

Let $K$ be an algebraically closed field. An algebraic monoid over $K$ is an irreducible variety $M$ such that the product map is a morphism of varieties. The set $G = G(M)$ of invertible elements of $M$ is an algebraic group. If $G$ is a reductive group, $M$ is called a reductive monoid.

Let $B \subseteq G$ be a Borel subgroup, $T \subseteq B$ a maximal torus, $W = N_G(T)/T$ the Weyl group of the pair $(G,T)$ and $S$ the set of simple reflections for $W$ corresponding to $B$, $\ell$ and $\leq$ the length function and the Bruhat-Chevalley order corresponding to $(W,S)$.

Recall that the Bruhat-Chevalley decomposition

$$G = \bigsqcup_{w \in W} B\dot{w}B, \text{ for } w = \dot{w}T \in N_G(T)$$

of the reductive group $G$ is controlled by the Weyl group $W$ of $G$, where $\dot{w}$ is any coset representative of $w \in W$.

Let $\overline{T}$ denote the Zariski closure of the maximal torus in $M$ and $E(\overline{T})$ the set of idempotents of $\overline{T}$. Then, the Weyl group $W$ of the pair $(G,T)$ and the set $E(\overline{T})$ of idempotents of the embedding $T \hookrightarrow M$ form a finite inverse semigroup $R = N_G(T)/T \cong W \cdot E(\overline{T})$ with unit group $W$ and the idempotent set $E(R) \cong E(\overline{T})$. The inverse semigroup $R$, which is called the Renner monoid of $M$, governs the Bruhat decomposition of the reductive monoid $M$:

$$M = \bigsqcup_{r \in R} B\dot{r}B, \text{ for } r = \dot{r}T \in N_G(T).$$

Recall that the Bruhat-Chevalley order for $(W,S)$ is defined by

$$x \leq y \text{ if and only if } BxB \subseteq ByB.$$ 

Similarly, on the Renner monoid $R$ of a reductive monoid $M$, the Bruhat-Chevalley-Renner order is defined by

$$\sigma \leq \tau \text{ if and only if } B\sigma B \subseteq B\tau B.$$  (2.1)
Observe that the poset structure on the Weyl group $W$ induced from the Renner monoid $R$ agrees with the original Bruhat poset structure on $W$.

Let $E(T)$ be the set of idempotent elements in the Zariski closure of the maximal torus $T$ in the monoid $M$. Similarly, denote the set of idempotents in the monoid $M$ by $E(M)$. One has $E(T) \subseteq E(M)$. There is a canonical partial order $\leq$ on $E(M)$, hence on $E(T)$, defined by

$$e \leq f \mbox{ if and only if } ef = e = fe. \quad (2.2)$$

Note that $E(T)$ is invariant under the conjugation action of the Weyl group $W$. We call a subset $\Lambda \subseteq E(T)$ a cross-section lattice if $\Lambda$ is a set of representatives for the $W$-orbits on $E(T)$. The bijection $\Lambda \to G \backslash M / G$ defined by $e \mapsto GeG$ is order preserving. Then, $\Lambda = \Lambda(B) = \{ e \in E(T) : Be = eBe \}$.

The decomposition $M = \bigsqcup_{e \in \Lambda} GeG$ of a reductive monoid into its $G \times G$ orbits has a counterpart for its Renner monoid $R$. Namely, the finite monoid $R$ can be written as a disjoint union

$$R = \bigsqcup_{e \in \Lambda} WeW, \quad (2.3)$$

parametrized by the cross-section lattice $\Lambda$.

For $e \in \Lambda$, define

$$W(e) := \{ x \in W : xe = ex \},$$

$$W_e := \{ x \in W : xe = e \} \trianglelefteq W(e).$$

Both $W(e)$ and $W_e$ are parabolic subgroups of $W$.

By $D(e)$ and $D_e$, denote the minimal coset representatives of $W(e)$ and $W_e$ respectively:

$$D(e) := \{ x \in W : x \mbox{ is of minimum length in } xW(e) \},$$

$$D_e := \{ x \in W : x \mbox{ is of minimum length in } xW_e \}.$$  

For any given element $\sigma \in WeW$, there exists unique $x \in D_e$ and $y \in D(e)$ so that $\sigma = xey^{-1}$. By definition, this form is unique and will be referred as the **standard form** for the element $\sigma$.

The length function for $R$ with respect to $(W,S)$ is defined as follows: Let $w_0$ and $v_0$ be the longest elements of $W$ and $W(e)$ respectively. Then
$w_0v_0$ is the longest element of $D(e)$. Set
\[
\ell(e) := \ell(w_0v_0) - \ell(v_0), \quad \text{and} \quad \ell(\sigma) := \ell(x) + \ell(e) - \ell(y).
\]
Note that in general, the length function $\ell$ does not necessarily agree with the rank function on the graded poset $(R, \leq)$. However, when restricted to a $J$-class $W e W \subseteq R$, the length function $\ell$ is the rank function on the induced poset $(W e W, \leq)$.

By [7], we know that given two elements $\theta$ and $\sigma$ with their standard forms $\theta = uev^{-1} \in W e W$ and $\sigma = xfy^{-1} \in W f W$,
\[
\theta \leq \sigma \iff e \leq f, \quad u \leq xw, \quad yw \leq v \quad \text{for some } w \in W(f)W_e. \quad (2.4)
\]

**Remark 2.1.** Let $M$ be an arbitrary monoid of Lie type. In [9], Putcha shows that $M$ has a Renner monoid as well as a cross section lattice. Moreover, all statements made in this section about Renner monoids and cross section lattices of reductive monoids are known to be true for those of $M$.

## 3 Orbit Hecke algebras

Let $M$ be a finite monoid of Lie type. We use the notation of the previous section.

In [20], Solomon constructs the Hecke algebra and the generic Hecke algebra for the monoid $M_n(\mathbb{F}_q)$ of $n \times n$ matrices over the finite field with $q$ elements, $\mathbb{F}_q$.

From now on, let $q$ be an indeterminate instead of a prime power. Following Solomon’s construction in [20], the generic Hecke algebra $\mathcal{H}(R)$ of the Renner monoid of $M$ is defined as follows:

The generic Hecke algebra $\mathcal{H}(R)$ is a $\mathbb{Z}[q^{-\frac{1}{2}}, q^{\frac{1}{2}}]$-algebra generated by a formal basis $\{A_\sigma\}_{\sigma \in R}$ with respect to multiplication rules
\[
A_s A_\sigma = \begin{cases} 
A_{s\sigma} & \text{if } \ell(s\sigma) = \ell(\sigma) + 1 \\
A_\sigma & \text{if } \ell(s\sigma) = \ell(\sigma) \\
q^{-1}A_{s\sigma} + (1 - q^{-1})A_\sigma & \text{if } \ell(s\sigma) = \ell(\sigma) - 1
\end{cases} \quad (3.1)
\]
\[
A_\nu A_\sigma = A_{\nu\sigma}
\]
for $s \in S, \quad \sigma, \nu \in R$, where $\ell(\nu) = 0$. The products $A_\sigma A_s$ are defined similarly.
Fix $e \in \Lambda$ and let $I \subseteq \mathcal{H}(R)$ be the two-sided ideal

$$I = I(e) = \bigoplus_{f < e, \sigma \in W_{fW}} \mathbb{Z}[q^{-\frac{1}{2}}, q^{\frac{1}{2}}]A_\sigma \subseteq \mathcal{H}(R). \quad (3.2)$$

The orbit Hecke algebra

$$\mathcal{H}(e) = \bigoplus_{\sigma \in W_{W}} \mathbb{Z}[q^{-\frac{1}{2}}, q^{\frac{1}{2}}]A_\sigma \quad (3.3)$$

is an ideal of $\mathcal{H}(R)/I$. Note that, if the idempotent $e \in \Lambda$ is the identity element $\text{id} \in W$ of the Weyl group, then the generic orbit Hecke algebra $\mathcal{H}(e)$ is isomorphic to generic Hecke algebra $\mathcal{H}(W)$ of $G$. The algebra $\hat{\mathcal{H}}(e) := \mathcal{H}(e) + \mathcal{H}(W)$ is called the augmented orbit Hecke algebra.

In [6], Kazhdan and Lusztig introduced an involution $\tau$ on the Hecke algebra $\mathcal{H}(W)$ with which they constructed two sets of polynomials $R$ and $P$, both indexed by a pair of Weyl group elements. Putcha [12] extends the Kazhdan-Lusztig involution $\tau$ to orbit Hecke algebras $\hat{\mathcal{H}}(e)$:

**Theorem 3.1** ([12], Theorem 4.1). There is a unique extension of the involution $\tau$ on $\mathcal{H}(W)$ to $\hat{\mathcal{H}}(e)$ such that for $e \in W_{eW}$ and $\sigma \in W_{W}$ in standard form $\sigma = \text{set}^{-1}$,

$$\overline{A}_\sigma := \sum_{z \in W_{W(e)}, y \in D(e)} \overline{R}_{z,y} A_{zey^{-1}}$$

$$\overline{A}_\sigma := q^{-\ell(t)} \overline{A}_\sigma \sum_{z \in W_{W(e)}, y \in D(e)} \overline{R}_{t_{z,y}} A_{zey^{-1}}. \quad (ii)$$

Here $R_{z,y}, R_{t_{z,y}} \in \mathbb{Z}[q]$ are $R$-polynomials of $W$.

**Corollary 3.2** ([12], Corollary 4.2). Let $\sigma \in W_{eW}$. Then there exists $R_{\theta,\sigma} \in \mathbb{Z}[q]$ for $\theta \in W_{eW}$, such that in $\hat{\mathcal{H}}(e)$,

(i) $\overline{A}_\sigma = q^{\ell(\sigma) - \ell(e)} \sum_{\theta \in W_{eW}} \overline{R}_{\theta,\sigma} A_\theta$

(ii) $R_{\theta,\sigma} \neq 0$ only if $\theta \leq \sigma$,

(iii) $R_{\theta,\theta} = 1$.

In Section 5, we answer the following question by Putcha:

**Problem 3.3** ([12], Problem 4.3). Determine the polynomials $R_{\theta,\sigma}$ explicitly for $\theta, \sigma \in W_{eW}$. Does $\theta \leq \sigma$ imply $R_{\theta,\sigma} \neq 0$?
4 Descent sets for the elements of a Renner monoid

An important ingredient in the study of the combinatorics of the Kazhdan-Lusztig theory for Weyl groups is the descent of an element \( w \in W \), which has been missing in the context of Renner monoids. In the following, we extend the notion of the descent set of an element \( w \in W \) to a \( J \)-class (a \( W \times W \)-orbit) in the Renner monoid.

Note that for a simple reflection \( s \) and \( \theta \in R \),
\[
s\theta < \theta \ (\text{resp., } =, >) \quad \text{if and only if} \quad \ell(s\theta) - \ell(\theta) = -1 \ (\text{resp., } 0, 1).
\]

The following lemma can be found in [14].

**Lemma 4.1.** Let \( I \subset S \), \( W_I \) the subgroup generated by \( I \), and \( D_I \) the minimal coset representatives of \( W/W_I \) in \( W \). Let \( x, y \in D_I \) and \( w, u \in W_I \).

(i) If \( xw < yu \), then there exist \( w_1, w_2 \in W \) satisfying \( w = w_1w_2 \) such that \( \ell(w) = \ell(w_1) + \ell(w_2) \), \( xw_1 \leq y \) and \( w_2 \leq u \).

(ii) If \( wx^{-1} < uy^{-1} \), then there exist \( w_1, w_2 \in W \) satisfying \( w = w_1w_2 \) such that \( \ell(w) = \ell(w_1) + \ell(w_2) \), \( w_1 \leq u \) and \( w_2x^{-1} \leq y^{-1} \).

**Corollary 4.2.** We use the notation of the previous Lemma. Let \( x \in D_I \) and \( s \in S \).

(i) If \( x < sx \), then either \( sx \in D_I \) or \( sx = xs' \) for some \( s' \in W_I \cap S \).

(ii) If \( sx < x \), then \( sx \in D_I \).

**Proof.** (i) Suppose \( sx = x's' \) for some \( x' \in D_I \) and \( s' \in W_I \). Let \( \text{id} \) denote the identity element of the Weyl group. We have
\[
x \cdot \text{id} = x(\text{id} \cdot \text{id}) \leq x's'
\]
and by the previous lemma, it follows that \( x \leq x' \). Therefore, \( l(x) \leq l(x') \).

On the other hand, since
\[
l(x) + 1 = l(sx) = l(x's') = l(x') + l(s'),
\]
we have either \( x = x' \) and \( s' \in I \), or \( s' = \text{id} \) and \( x' = sx \).
(ii) If $W$ is finite and $W_I \subseteq W$, then Björner and Wachs show in [3, Theorem 4.1] that $D_I$, which is a generalized quotient, is a lower interval of the weak Bruhat order on $W$. Hence, the result follows.

For $\sigma \in W e W$, we define the left descent set and the right descent set with respect to $S$ as

$$\text{Des}_L(\sigma) = \{s \in S \mid \ell(s\sigma) < \ell(\sigma)\} \text{ and } \text{Des}_R(\sigma) = \{s \in S \mid \ell(\sigma s) < \ell(\sigma)\}.$$ 

By Corollary 4.2, we reformulate the descent sets as follows:

**Lemma 4.3.** Suppose $\sigma \in W e W$ has the standard form $x e y^{-1}$, where $x \in D_e$ and $y \in D(e)$. Then,

- $\text{Des}_L(\sigma) = \{s \in S \mid \ell(sx) < \ell(x)\}$, and
- $\text{Des}_R(\sigma) = \{s \in S \mid \ell(sy) > \ell(y), \text{ and either } sy \in D(e), \text{ or } sy = ys' \text{ for some } s' \in W(e) \cap S \text{ and } \ell(xs') < \ell(x)\}$.

**Remark 4.4.** Let $\nu \in W e W$ be the unique element so that $\ell(\nu) = 0$. Then, it is easy to see that both descent sets of $\nu$ are empty. It is essential to emphasize that unlike the usual Weyl group setting, not every $\sigma \in W e W$ has a left descent. On the other hand, by using [3, Theorem 4.1], one can show the following.

**Corollary 4.5.** The union $\text{Des}_L(\sigma) \cup \text{Des}_R(\sigma)$ is non-empty for any $\sigma \in R$ if $\ell(\sigma) \neq 0$.

The following example illustrates the possible cases for the descent sets of $\sigma \in W e W$ for $W = S_n$.

**Example 4.6.** Let $M_4(\mathbb{F}_q)$ be the finite monoid of $4 \times 4$ matrices over the finite field $\mathbb{F}_q$ with $q$ elements. The Renner monoid of $M_4(\mathbb{F}_q)$ consists of all $4 \times 4$ partial permutation matrices, and its Weyl subgroup is the symmetric group $W = S_4$ consisting of permutation matrices. Given a matrix $x = (x_{ij})$ in the Renner monoid, let $(a_1 a_2 a_3 a_4)$ be the sequence defined by

$$a_j = \begin{cases} 
0, & \text{if the } j\text{th column consists of zeros;} \\
i, & \text{if } x_{ij} = 1.
\end{cases} \tag{4.1}$$

A partial permutation matrix is a $0 - 1$ matrix with at most one 1 in each row and each column.
For example, the sequence associated with the matrix
\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]
is (3040). Let \( e \) be the idempotent \( e = (1200) \in WeW \). Then, \( W(e) \cong S_2 \times S_2 \). The table below illustrates the possible cases for the descent sets for some choices of \( \sigma \in WeW \):

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>( \text{Des}_L )</th>
<th>( \text{Des}_R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1234) e(3412) = (0012)</td>
<td>( \emptyset )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>(1324) e(3412) = (0013)</td>
<td>{s_2}</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>(1234) e(1342) = (1002)</td>
<td>( \emptyset )</td>
<td>{s_1}</td>
</tr>
<tr>
<td>(3214) e(1342) = (3002)</td>
<td>{s_1, s_2}</td>
<td>{s_1}</td>
</tr>
<tr>
<td>(4213) e(3124) = (0420)</td>
<td>{s_1, s_3}</td>
<td>{s_2, s_3}</td>
</tr>
</tbody>
</table>

Here, \( s_1 = (2134) \), \( s_2 = (1324) \), \( s_3 = (1243) \) are the simple reflections for \( S_4 \).

5 \quad \text{\textbf{R-polynomials}}

Given an interval \([\theta, \sigma] \subseteq WeW\), define its length \( \ell(\theta, \sigma) := \ell(\sigma) - \ell(\theta) \) and its \( R \)-polynomial \( R_{\theta, \sigma}(q) \) as in Corollary 3.2.

\textbf{Theorem 5.1.} Let \( \sigma, \theta \in R \) be so that \( \ell(\sigma) \neq 0 \) and \( \theta \leq \sigma \). Then, for \( s \in \text{Des}_L(\sigma) \), one has

\[
R_{\theta, \sigma} = \begin{cases}
R_{s\theta, s\sigma} & \text{if } s\theta < \theta, \\
qR_{s\theta, s\sigma} & \text{if } s\theta = \theta, \\
(q-1)R_{\theta, s\sigma} + qR_{s\theta, s\sigma} & \text{if } s\theta > \theta.
\end{cases}
\]

Otherwise, there exists \( s \in \text{Des}_R(\sigma) \) such that

\[
R_{\theta, \sigma} = \begin{cases}
R_{\theta s, \sigma s} & \text{if } \theta s < \theta, \\
qR_{\theta s, \sigma s} & \text{if } \theta s = \theta, \\
(q-1)R_{\theta, \sigma s} + qR_{\theta s, \sigma s} & \text{if } \theta s > \theta.
\end{cases}
\]

Moreover, if \( \theta s > \theta \) and \( ss\sigma = \sigma \), then \( R_{\theta, \sigma} = qR_{s\theta, s\sigma} \).
For any \( \theta \), we have \( R_{\theta,\theta}(q) = 1 \). If \([\theta, \sigma]\) is an interval of length 1 in \( W e W \), then \( R_{\theta,\sigma}(q) = q - 1 \).

**Remark 5.2.** Given two elements \( u, v \) of a Weyl group \( W \), the polynomial \( R_{u,v}(q) \neq 0 \) if and only if \( u \leq v \). If \( u \leq v \), then \( R_{u,v}(q) \) is a monic polynomial of degree \( \ell(u, v) \) whose constant term is \((-1)^{\ell(u, v)}\).

The following proposition answers Problem 3.3 for the orbit Hecke algebras.

**Proposition 5.3.** Let \( \theta \leq \sigma \in W e W \). Then, \( R_{\theta,\sigma} \) is a monic polynomial of degree \( \ell(\theta, \sigma) \) whose the constant term is either 0 or \((-1)^{\ell(\theta, \sigma)}\). In particular, 

\[
R_{\theta,\sigma} \neq 0 \quad \text{if and only if} \quad \theta \leq \sigma.
\]

**Proof.** We prove the statement about the constant term. The statement about the degree and the leading term is proved similarly, via induction on \( \ell(\sigma) \).

Clearly, the constant term statement holds if \( \ell(\sigma) \leq 1 \). As the induction hypothesis, we assume that for all pairs \( \rho, \tau \) in \( W e W \) with \( \ell(\tau) < \ell(\sigma) \), the constant term of \( R_{\rho,\tau}(q) \) is either 0 or \((-1)^{\ell(\rho, \tau)}\).

Let \( \theta \in W e W \) so that \( \theta \leq \sigma \). If \( R_{\theta,\sigma}(0) = 0 \), there is nothing to prove.

Assume that the constant term of \( R_{\theta,\sigma} \) is non-zero. Without loss of generality, we assume that there exists \( s \in \text{Des}_L(\sigma) \). Then, by Theorem 5.1, we must have either \( s\theta > \theta \), or \( s\theta < \theta \).

First suppose that \( s\theta < \theta \). Then \( R_{\theta,\sigma}(q) = R_{s\theta,s\sigma}(q) \). Since \( \ell(s\theta, s\sigma) = \ell(\theta, \sigma) \), \( R_{\theta,\sigma}(0) \) equals \((-1)^{\ell(\theta, \sigma)}\).

Next, suppose that \( s\theta > \theta \). Therefore, \( R_{\theta,\sigma} = (q - 1)R_{\theta,s\sigma} + qR_{s\theta,s\sigma} \). Note that \( \theta \leq s\sigma \). Consequently, \( R_{\theta,\sigma}(0) = -R_{\theta,s\sigma}(0) \). Hence, the latter is nonzero. Furthermore, by the induction hypothesis, it is equal to \((-1)^{\ell(\theta, s\sigma)}\). Therefore, \( R_{\theta,\sigma}(0) = (-1)^{\ell(\theta, \sigma)} \) as claimed. \( \Box \)

Let \( q_\sigma \) and \( \varepsilon_\sigma \) denote \( q^{\ell(\sigma)} \) and \((-1)^{\ell(\sigma)}\) respectively for \( \sigma \in W e W \).

Arguing as in the proof of the theorem above, one can show that if \( R_{\theta,\sigma}(q) \) has a non-zero constant term, then

\[
\overline{R_{\theta,\sigma}} = \varepsilon_\theta \varepsilon_\sigma q_\theta q_\sigma^{-1} R_{\theta,\sigma}(q).
\]

However, this equality fails if \( R_{\theta,\sigma}(0) = 0 \).

The lifting property for Weyl groups states that, given \( u < v \) in \( W \) and a simple reflection \( s \), if \( u > su \) and \( v < sv \), then \( u < sv \) and \( su < v \). We will use the lifting property for \( W \) to prove the orbits \( W e W \) have the lifting property.
Corollary 5.4 (Lifting Property for WeW). Let $\theta = uev^{-1}$ and $\sigma = xey^{-1}$ be in standard form, $\theta < \sigma$ and $s$ be a simple reflection.

(a) If $\theta < s\theta$ and $\sigma < s\sigma$, then $s\theta < s\sigma$.
(b) If $s\theta \geq \theta$ and $s\sigma \leq \sigma$, then $\theta \leq s\sigma$ and $s\theta \leq \sigma$.

Proof. To make the matters short, use Theorem 5.1 and Proposition 5.3 to prove (a) or (b), if any of the inequalities is an equality. What remains to be shown is (b) in the strict case: $s\theta > \theta$ and $s\sigma < \sigma$.

Because $s\sigma < \sigma$, by Lemma 4.3, $sx < x$ and by Corollary 4.2 (b), $sx \in D_e$. Thus, the standard form for $s\sigma$ is $(sx)e^{-1}$.

Since $s\theta > \theta$, we observe that $su \not\leq u$. Otherwise, $(su)e^{-1}$ is the standard form of $s\theta$ resulting in a contradiction: $\ell(s\theta) < \ell(\theta)$.

As $\theta \leq \sigma$, there is $w \in W(e)$ so that $u \leq xw$ and $yw \leq v$. First, we prove $\theta \leq s\sigma$.

- Case $xw \leq s(xw)$: Then $u \leq xw \leq sxw$. Because $(sx)e^{-1}$ is the standard form of $s\sigma$, we conclude that $\theta \leq s\sigma$.

- Case $s(xw) \leq xw$: Apply the lifting property for $W$ to $u$ and $xw$. So, $u \leq s(xw) = (sx)w$ and again, we get $\theta \leq s\sigma$.

To prove $s\theta \leq \sigma$, apply (a) to the pair $\theta < s\sigma$.

Proposition 5.5. For all $\theta, \sigma \in WeW$,

$$\sum_{\theta \leq \nu \leq \sigma} R_{\theta, \nu} q_\nu^{-1} R_{\nu, \sigma} = \delta_{\theta, \sigma}. \quad (5.1)$$

We call an interval $[\theta, \sigma]$ linear, if the interval $[\theta, \sigma]$ is totally ordered with respect to Bruhat-Chevalley-Renner order. In this case, the interval $[\theta, \sigma]$ has $\ell(\theta, \sigma) + 1$ elements.

Using the above proposition in conjunction with Proposition 5.3, one can classify length 2 intervals $[\theta, \sigma] \subset WeW$ with respect to their $R$-polynomials, or equivalently, with respect to the constant terms of their $R$-polynomials:

<table>
<thead>
<tr>
<th>Interval</th>
<th>Num. of Elements</th>
<th>$[\theta, \sigma] \subset M_4(F_q)$</th>
<th>$R_{\theta, \sigma}(q)$</th>
<th>$R_{\theta, \sigma}(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>linear</td>
<td>3</td>
<td>$(0001) &lt; (0003)$</td>
<td>$q(q - 1)$</td>
<td>0</td>
</tr>
<tr>
<td>diamond</td>
<td>4</td>
<td>$(0012) &lt; (0023)$</td>
<td>$(q - 1)^2$</td>
<td>1</td>
</tr>
</tbody>
</table>

(5.2)
The Möbius function $\mu_{\theta, \sigma}$ of the interval $[\theta, \sigma]$ is defined by

$$
\mu_{\theta, \sigma} := \begin{cases} 
0 & \text{if } \theta \not\leq \sigma, \\
1 & \text{if } \theta = \sigma, \\
-\sum_{\theta \leq \tau < \sigma} \mu_{\theta, \tau} & \text{if } \theta < \sigma.
\end{cases}
$$

(5.3)

**Corollary 5.6.** For $\theta, \sigma \in \text{WeW}$,

$$
\mu_{\theta, \sigma} = R_{\theta, \sigma}(0).
$$

**Proof.** The equality is clear for $\theta \not\leq \sigma$ and $\theta = \sigma$. For $\theta < \sigma$, evaluating (5.1) at $q = 0$ yields

$$
R_{\theta, \sigma}(0) = -\sum_{\theta \leq \tau < \sigma} R_{\theta, \tau}(0).
$$

Thus, $\mu_{\theta, \sigma} = R_{\theta, \sigma}(0)$. □

Putcha conjectures (Conjecture 2.7 [15]) the following for reductive monoids, a subclass of monoids of Lie type:

$$
\mu_{\theta, \sigma} = \begin{cases} 
(-1)^{\ell(\sigma) - \ell(\theta)} & \text{if every length 2 interval in } [\theta, \sigma] \text{ has 4 elements,} \\
0 & \text{otherwise.}
\end{cases}
$$

In the next section we prove that

**Theorem 5.7.** Putcha’s conjecture holds for the $J$-classes in the monoids of Lie type.

To prove this theorem, we examine the interplay between the $R$-polynomial $R_{\theta, \sigma}$ of an interval $[\theta, \sigma]$ and the subintervals $[\alpha, \beta]$ contained in $[\theta, \sigma]$, especially the subintervals of length 2.

Our next result is about the relationship between $R_{\theta, \sigma}(0)$ and $R_{\alpha, \beta}(0)$ for some subinterval $[\alpha, \beta]$ of $[\theta, \sigma]$.

**Proposition 5.8.** Given an interval $[\theta, \sigma]$ with $R_{\theta, \sigma}(0) \neq 0$, then

(a) If $s\sigma < \sigma$ (or, $s\theta > \theta$, $\sigma s < \sigma$, $\theta s > \theta$) for some simple reflection $s$, then for any $\alpha \in [\theta, \sigma]$, $s\alpha \neq \alpha$.

(b) For a subinterval $[\alpha, \beta]$ of $[\theta, \sigma]$ in WeW, $R_{\alpha, \beta}(0) \neq 0$. 

12
In the proof, we show (a) only for the case \( s \sigma < \sigma \). The other cases are virtually the same.

**Proof.** We proceed by induction on \( \ell(\sigma) \). If \( \ell(\sigma) \leq 1 \), then \( R_{\theta,\sigma}(q) = (q - 1)^{\ell(\theta,\sigma)} \) and both statements hold trivially.

**Induction step.** Note that \( s \theta \neq \theta \). If \( s \theta < \theta \), then one can apply induction to \([s \theta, s \sigma]\) as \( R_{s \theta, s \sigma}(0) = R_{\theta, \sigma}(0) \neq 0 \). If \( s \theta > \theta \), then one can apply induction to \([\theta, s \sigma]\) as \( R_{\theta, s \sigma}(0) = -R_{\theta, \sigma}(0) \neq 0 \).

Let \( \rho := \min\{\theta, s \theta\} \). By definition, \( s \rho > \rho \) and by lifting property, for any \( \alpha \in [\theta, \sigma] \), \( \rho \leq \min\{\alpha, s \alpha\} \). In short, we can apply induction to \([\rho, s \sigma]\), since \( \ell(s \sigma) < \ell(\sigma) \).

(a) \( R_{\theta, \sigma}(0) \neq 0 \) and a simple reflection \( s \) so that \( s \sigma < \sigma \) is given. If for all \( \alpha \in [\theta, \sigma] \), \( s \alpha \neq 0 \), there is nothing to prove.

Assume that for some \( \alpha \in [\theta, \sigma] \), \( s \alpha = \alpha \). By the lifting property, \( \alpha \leq s \sigma \) and hence \( \alpha \in [\rho, s \sigma] \). Use induction to conclude that \( s \alpha \neq \alpha \) as \( s \rho > \rho \). This results in a contradiction. Therefore, we conclude that there is no \( \alpha \in [\theta, \sigma] \) such that \( s \alpha = \alpha \).

(b) First, assume that \( s \sigma < \sigma \) for some \( s \in S \).

Pick any interval \([\alpha, \beta]\) in \([\theta, \sigma]\). If \( s \beta > \beta \), by the lifting property \([\alpha, \beta] \subset [\rho, s \sigma]\) (apply the induction step).

Otherwise, \( s \beta < \beta \). Then, there are two cases:

- \( s \alpha < \alpha \): Then, \( R_{\alpha, \beta}(0) = R_{s \alpha, s \beta}(0) \) and \([s \alpha, s \beta] \subset [\rho, s \sigma]\) (apply the induction step).
- \( s \alpha > \alpha \): Then, \( R_{\alpha, \beta}(0) = -R_{\alpha, s \beta}(0) \) and \([\alpha, s \beta] \subset [\rho, s \sigma]\) (apply the induction step).

Say \( R_{\alpha, \beta}(0) \neq 0 \) for all proper subintervals \([\alpha, \beta]\) of a given interval \([\theta, \sigma]\). Does this imply \( R_{\theta, \sigma}(0) \neq 0 \)?

Unfortunately, this is even false for a linear length 2 interval \([\theta, \sigma]\). Any proper subinterval \([\alpha, \beta]\) is of length \( \leq 1 \), hence \( R_{\alpha, \beta}(0) \neq 0 \), yet \( R_{\theta, \sigma}(0) = 0 \).

### 6 Intervals of length \( \leq 2 \)

It is clear that an interval \([\theta, \sigma]\) in \( W e W \) with \( R_{\theta, \sigma}(0) = 0 \) can never be embedded into some Weyl group \( W' \) as a subinterval \([u, v]\) so that the \( R_{\theta, \sigma}(q) \) equals \( R_{u, v}(q) \) for the simple reason that \( R_{u, v}(0) = (-1)^{\ell(u, v)} \neq 0 \). In fact,
more is true. We prove the following fact about the Bruhat graphs of the intervals $[\theta, \sigma]$ with $R_{\theta, \sigma}(0) = 0$:

**Theorem 6.1.** An interval $[\theta, \sigma]$ in $WeW$ cannot be embedded into any Weyl group $W'$ as a subinterval if $R_{\theta, \sigma}(0) = 0$.

We prove this assertion by showing that

**Proposition 6.2.** Given an interval $[\theta, \sigma]$ in $WeW$, the polynomial $R_{\theta, \sigma}(q)$ vanishes for $q = 0$ if and only if there exists a linear length 2 interval $[\alpha, \beta]$ inside $[\theta, \sigma]$.

Theorem 6.1 follows from this proposition because of the well-known fact that any interval $[u, v]$ of length 2 in a Weyl group $W'$ is diamond shaped and contains 4 elements.

**Lemma 6.3.** Given an interval $[\theta, \sigma]$ in $WeW$, if there exists a simple reflection $s \in S$ such that $s\sigma < \sigma$ (or, $s\theta > \theta$, $s\sigma > \sigma$, $\theta s > \theta$) and $s\rho = \rho$ (resp. $\rho s = \rho$) for some $\rho \in [\theta, \sigma]$, then $[\theta, \sigma]$ contains a linear length 2 interval.

**Proof.** We prove the lemma only for $s\sigma < \sigma$ as all the other cases are proved essentially the same way.

By assumption, the set $\{\rho : s\rho = \rho\}$ is non-empty. Pick a maximal element $\alpha$ in this set. Then, for all $\beta \in [\theta, \sigma]$ with $\alpha < \beta$, $s\beta \neq \beta$.

Now pick an element $\beta$ from the interval $[\alpha, \sigma]$ covering $\alpha$. By the choice of $\alpha$, the element $s\beta$ differs from $\beta$ as indicated above. Because $\beta$ covers $\alpha$, we have $R_{\alpha, \beta}(q) = q - 1$.

If it were that $s\beta < \beta$, then by the recurrence relations of Theorem 5.1, $q - 1 = qR_{s\alpha, s\beta}(q)$, which implies that $\alpha = s\beta$ and $q - 1 = q$, both being obvious contradictions.

Therefore, $s\beta > \beta$ and $R_{\alpha, s\beta}(0) = 0$. By lifting property, $s\beta \leq \sigma$. The subinterval $[\alpha, s\beta]$ is a linear length 2 interval in $[\theta, \sigma]$ as required.

**Lemma 6.4.** Let $[\theta, \sigma]$ be an interval which contains a linear length 2 interval $[\alpha, \beta]$. Say for some $s \in S$, $\theta < s\theta$ and $\sigma < s\sigma$. Then, the interval $[s\theta, s\sigma]$ contains a linear length 2 interval.

**Proof.** Say $s\alpha \leq \alpha$. By lifting property, $s\theta \leq \alpha < \beta < \sigma < s\sigma$. The interval $[\alpha, \beta] \subset [s\theta, s\sigma]$ is the required one.

Otherwise, $s\alpha > \alpha$. Since $[\alpha, \beta]$ is a linear length 2 interval, $s\beta \geq \beta$. If $\beta > s\beta$, then $R_{s\alpha, s\beta} = R_{\alpha, \beta}$ and $s\theta \leq s\alpha < s\beta < s\sigma$. 

14
If \( s\beta = \beta \), then by lifting property \( s\theta \leq \beta \leq \sigma < s\sigma \). The results follows from Lemma 6.3.

**Proof of Proposition 6.2.** \((\iff)\) If there is such \( \alpha, \beta \), then \( R_{\alpha,\beta}(0) = 0 \) and the rest follows from Proposition 5.8.

\((\Rightarrow)\) Prove by induction on the length \( \ell(\sigma) \).

The base case is \( \ell(\theta) = 0 \) and \( \ell(\sigma) = 2 \) which follows from (5.2). As usual, assume that \( s\sigma < \sigma \) for a simple reflection \( s \in S \).

If \( s\theta = \theta \), the result follows by Lemma 6.3.

If \( s\theta > \theta \), then \( R_{\theta,\sigma} = (q - 1)R_{\theta,s\sigma} + qR_{s\theta,s\sigma} \). Hence \( 0 = R_{\theta,\sigma}(0) = R_{\theta,s\sigma}(0) \). The length of the interval \([\theta, s\sigma]\) is \( \ell(\theta, \sigma) - 1 \). Apply induction.

If \( s\theta < \theta \), then apply the Lemma above and then use induction. This finishes the proof.

**Proof of Theorem 5.7.** We reiterate what we have already shown.

For a given interval \([\theta, \sigma]\) in \( W e W \), the following are equivalent:

1. \( \mu_{\theta,\sigma} = R_{\theta,\sigma}(0) \neq 0 \),
2. \( \mu_{\theta,\sigma} = R_{\theta,\sigma}(0) = (-1)^{\ell(\theta,\sigma)} \),
3. All length 2 subintervals of \([\theta, \sigma]\) have 4 elements (and are diamond shaped).

Otherwise, \( \mu_{\theta,\sigma} = R_{\theta,\sigma}(0) = 0 \) and \([\theta, \sigma]\) contains a length 2 subinterval with 3 elements (a linear length 2 subinterval).

**References**


