1. INTRODUCTION.

Symmetric functions \( \{E_{n,k}(X)\}_{k=1}^n \), defined by the Newton interpolation

\[
e_n[X \frac{1-z}{1-q}] = \sum_{k=1}^n (z; q)_k \frac{E_{n,k}(X)}{(q; q)_k}
\]

plays an important role in the Garsia-Haglund proof of the \( q,t \)-Catalan conjecture, [2].

Let \( \Lambda^n_{Q(q,t)} \) be the space of symmetric functions of degree \( n \), over the field of rational functions \( Q(q,t) \), and let \( \nabla : \Lambda^n_{Q(q,t)} \to \Lambda^n_{Q(q,t)} \) be the Garsia-Bergeron operator.

By studying recursions, Garsia and Haglund show that the coefficient of the elementary symmetric function \( e_n(X) \) in the image \( \nabla(E_{n,k}(X)) \) of \( E_{n,k}(X) \) is equal to the following combinatorial summation

\[
\langle \nabla(E_{n,k}(X)), e_n(X) \rangle = \sum_{\pi \in D_{n,k}} q^{\text{area}(\pi)} t^{\text{bounce}(\pi)},
\]

where \( D_{n,k} \) is the set of all Dyck paths with initial \( k \) North steps followed by an East step. Here \( \text{area}(\pi) \) and \( \text{bounce}(\pi) \) are two numbers associated with a Dyck path \( \pi \). It is conjectured in [4], more generally, that the \( \nabla E_{n,k}(X) \) are “Schur positive.”

In [1], using (1.1), Can and Loehr prove the \( q, t \)-Square conjecture of the Loehr and Warrington [7].

The aim of this article is to understand the functions \( \{E_{n,k}(X)\}_{k=1}^n \) better. We prove that the vector subspace generated by the set \( \{E_{n,k}(X)\}_{k=1}^n \) of the space \( \Lambda^n_{Q(q)} \) of degree \( n \) symmetric functions over the field \( Q(q) \), is equal to the subspace generated by

\[
\{s_{(k,1^{n-k})}[X/(1-q)]\}_{k=1}^n,
\]

Schur functions of hook shape, plethystically evaluated at \( X/(1-q) \).

In particular, we determine explicitly the transition matrix and its inverse from \( \{E_{n,k}(X)\}_{k=1}^n \) to \( \{s_{(k,1^{n-k})}[X/(1-q)]\}_{k=1}^n \). The entries of the matrix turns out to be cocharge Kostka-Foulkes polynomials.
We find the expansion of $E_{n,k}(X)$ into the Hall-Littlewood basis, and as a corollary we recover a closed formula for the cocharge Kostka-Foulkes polynomials $\tilde{K}_{\lambda,\mu}(q)$ when $\lambda$ is a hook shape:

$$\tilde{K}_{(n-k,1^k)}(q) = (-1)^k \sum_{i=0}^{k} (-1)^i q^{i} \left[ \begin{array}{c} r \\ i \end{array} \right].$$

Here, $\mu$ is a partition of $n$ whose first column is of height $r$.

2. **Background.**

2.0.1. **Notation.** A partition $\mu$ of $n \in \mathbb{Z}_{>0}$, denoted $\mu \vdash n$, is a nonincreasing sequence $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_k > 0$ of numbers such that $\sum \mu_i = n$. The conjugate partition $\mu' = \mu'_1 \geq \ldots \geq \mu'_s > 0$ is defined by setting $\mu'_i = |\{ \mu_r : \mu_r \geq i \}|$.

$Par(n,r)$ denotes the set of all partitions $\mu \vdash n$ whose biggest part is equal to $\mu_1 = r$.

We identify a partition $\mu$ with its Ferrers diagram, in French notation. Thus, if the parts of $\mu$ are $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_k > 0$, then the corresponding Ferrers diagram have $\mu_i$ lattice cells in the $i^{th}$ row (counting from bottom to up).

Following Macdonald, [8] the arm, leg, coarm and coleg of a lattice square $s$ are the parameters $a(\mu)(s), l(\mu)(s), a'(\mu)(s)$ and $l'(\mu)(s)$ giving the number of cells of $\mu$ that are respectively strictly EAST, NORTH, WEST and SOUTH of $s$ in $\mu$.

Given a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$, we set

\begin{equation}
(2.1) \quad n(\mu) = \sum_{i=1}^{k} (i - 1) \mu_i = \sum_{s \in \mu} l(\mu)(s).
\end{equation}

We also set

\begin{equation}
(2.2) \quad h(\mu)(q,t) = \prod_{s \in \mu} (q^{a(\mu)(s)} - t^{l(\mu)(s)} + 1) \quad \text{and} \quad \tilde{h(\mu)}(q,t) = \prod_{s \in \mu} (t^{l'(\mu)(s)} - q^{a'(\mu)(s)} + 1).
\end{equation}

Let $F$ be a field, and let $X = \{x_1, x_2, \ldots\}$ be an alphabet (a set of indeterminates). The algebra of symmetric functions over $F$ with the variable set $X$ is denoted by $\Lambda_F(X)$.

If $\mathbb{Q} \subseteq F$, it is well known that $\Lambda_F(X)$ is freely generated by the set of power-sum symmetric functions

$$\{p_r(X) : r = 1, 2, \ldots \text{ and } p_r(X) = x_1^r + x_2^r + \cdots \}.$$

The algebra, $\Lambda_F(X)$ has a natural grading (by degree).

$$\Lambda_F(X) = \bigoplus_{n \geq 0} \Lambda_F^n(X),$$

where $\Lambda_F^n(X)$ is the space of homogenous symmetric functions of degree $n$. 
A basis for the vector space $\Lambda^n F(X)$ is given by the set $\{p_\mu(X)\}_{\mu \vdash n}$,

$$p_\mu(X) = \prod_{i=1}^k p_{\mu_i}(X), \text{ where } \mu = \sum_{i=1}^k \mu_i. \tag{2.3}$$

Another basis for $\Lambda^n F(X)$ is given by the Schur functions $\{s_\mu(X)\}_{\mu \vdash n}$, where $s_\mu(X)$ is defined as follows. Let

$$e_n(X) = \sum_{1 \leq i_1 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n} \tag{2.4}$$

be the $n$'th elementary symmetric function. If $\mu = \sum_{i=1}^k \mu_i$, then

$$s_\mu(X) = \det(e_{\mu'_i-i+j}(X))_{1 \leq i,j \leq m}, \tag{2.5}$$

where $\mu'_i$ is the $i$'th part of the conjugate partition $\mu' = (\mu'_1, \ldots, \mu'_l)$ and $m \geq l$.

### 2.0.2. Plethysm

For the purposes of this section, we represent an alphabet $X = \{x_1, x_2, \ldots\}$ as a formal sum $X = \sum x_i$. Thus, if $Y = \sum y_i$ is another alphabet, then

$$XY = \left(\sum x_i\right)\left(\sum y_i\right) = \sum_{i,j} x_i y_j = \{x_i y_j\}_{i,j \geq 1}, \tag{2.6}$$

and

$$X + Y = \left(\sum_i x_i\right) + \left(\sum_j y_j\right) = \{x_i, y_j\}_{i,j \geq 1}. \tag{2.7}$$

The formal additive inverse, denoted $-X$, of an alphabet $X = \sum x_i$ is defined so that $-X + X = 0$.

In this vein, if $p_k(X) = \sum_{k \geq 1} x_i^k$ is a power sum symmetric function, we define

$$p_k[XY] = p_k[X]p_k[Y], \tag{2.8}$$

$$p_k[X + Y] = p_k[X] + p_k[Y], \tag{2.9}$$

$$p_k[-X] = -p_k[X]. \tag{2.10}$$

This operation is called plethysm. Since $\Lambda^0 F$ is freely generated by the power sums, the plethysm operator can be extended to the other symmetric functions.

In fact, using plethysm, one defines the following bases for $\Lambda^n_{Q(q)}$ and $\Lambda^n_{Q(q,t)}$, respectively.

**Theorem-Definition 1.** (cocharge Hall-Littlewood polynomials)

There exists a basis $\{\tilde{H}_\mu(X; q)\}_{\mu \vdash n}$ for the vector space $\Lambda^n_{Q(q)}$, which is uniquely characterized by the properties

1. $\tilde{H}_\mu(X; q) \in \mathbb{Z}[q]\{s_\lambda : \lambda \geq \mu\}$,
2. $\tilde{H}_\mu((1 - q)X; q) \in \mathbb{Z}[q]\{s_\lambda : \lambda \geq \mu'\}$,
3. $\langle \tilde{H}_\mu(X; q), s_{(n)} \rangle = 1$. 


Theorem-Definition 2. (Modified Macdonald polynomials)
There exists a basis \{\tilde{H}_\mu(X; q, t)\}_{\mu \vdash n} for the vector space \Lambda^n_{Q(q,t)}, which is uniquely characterized by the properties

1. \tilde{H}_\mu(X; q, t) \in \mathbb{Z}[q]\{s_\lambda : \lambda \geq \mu\},
2. \tilde{H}_\mu[(1 - q)X; q, t] \in \mathbb{Z}[q]\{s_\lambda : \lambda \geq \mu'\},
3. \langle \tilde{H}_\mu[(1 - t); q, t], s_{(n)} \rangle = 1.

It follows from these Theorem-Definitions that

\begin{equation}
\tilde{H}_\mu(X; 0, t) = \tilde{H}_\mu(X; t), \tag{2.11}
\end{equation}

\begin{equation}
\tilde{H}_\mu(X; q, t) = \tilde{H}_\mu'(X; t, q). \tag{2.12}
\end{equation}

2.0.3. Kostka-Foulkes and Kostka-Macdonald polynomials. Let

\begin{align*}
\tilde{H}_\mu(X; q) &= \sum_\lambda \tilde{K}_{\lambda\mu}(q)s_\lambda, \\
\tilde{H}_\mu(X; q, t) &= \sum_\lambda \tilde{K}_{\lambda\mu}(q, t)s_\lambda
\end{align*}

be, respectively, the Schur basis expansions of the Hall-Littlewood and Macdonald symmetric functions. The coefficients of the Schur functions are called, respectively, the cocharge Kostka-Foulkes polynomials, and the modified Kostka-Macdonald polynomials. It is known that \tilde{K}_{\lambda\mu}(q, t), \tilde{K}_{\lambda\mu}(q) \in \mathbb{N}[q, t].

It follows from equation (2.11) and the Schur basis expansions that

\begin{equation}
\tilde{K}_{\lambda\mu}(0, t) = \tilde{K}_{\lambda\mu}(t). \tag{2.13}
\end{equation}

2.0.4. Cauchy Identities. Let \( X = \sum x_i \) be an alphabet, and let

\[ \Omega[X] = \exp\left(\sum_{k=1}^{\infty} \frac{p_k(X)}{k}\right). \]

Then,

\begin{equation}
\Omega[X] = \prod_i \frac{1}{1 - x_i} = \sum_{n=0}^{\infty} s_n(X), \tag{2.14}
\end{equation}

\begin{equation}
\Omega[X] = \prod_i (1 - x_i) = \sum_{n=0}^{\infty} s_{1^n}(X). \tag{2.15}
\end{equation}

If \( Y = \sum y_i \) is another alphabet, then

\begin{equation}
\epsilon_n[XY] = \sum_{\mu \vdash n} s_\mu[X]s_{\mu'}[Y], \tag{2.16}
\end{equation}

\begin{equation}
\epsilon_n[XY] = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu[X; q, t]\tilde{H}_\mu[Y; q, t]}{\tilde{h}_\mu(q, t)\tilde{h}'_\mu(q, t)}, \tag{2.17}
\end{equation}

where \( \tilde{h}_\mu(q, t) \) and \( \tilde{h}'_\mu(q, t) \) are as in (2.2).
(2.18) \[ s_\mu[1 - z] = \begin{cases} (-z)^k(1 - z) & \text{if } \mu = (n - k, 1^k), \\ 0 & \text{otherwise}. \end{cases} \]

2.0.5. Cauchy’s \(q\)-binomial theorem. Let \((z; q)_k = (1 - z)(1 - qz) \cdots (1 - q^{k-1}z)\), and let \(\left[ \begin{array}{c} k \\ r \end{array} \right] = \frac{(q; q)_k}{(q; q)_r(q; q)_{k-r}}.\)

Then, the Cauchy \(q\)-binomial theorem states that

\[(z; q)_k = \sum_{r=0}^{k} z^r (-1)^r e_r[1, q, ..., q^{k-1}] = \sum_{r=0}^{k} z^r q^{2r} (-1)^r \left[ \begin{array}{c} k \\ r \end{array} \right].\]

### 3. Symmetric functions \(E_{n,k}(X)\).

The family \(\{E_{n,k}(X)\}_{k=1}^{n}\) of symmetric functions are defined by the plethystic identity

\[(3.1) \quad e_n[X^{(1 - z)/(1 - q)}] = \sum_{k=1}^{n} (z; q)_k E_{n,k}(X).\]

Let \(0 \leq k \leq r\), and let

\[(3.2) \quad T_{k+1,r} = (-1)^k \sum_{i=0}^{k} (-1)^i q^i \left[ \begin{array}{c} r \\ i \end{array} \right].\]

**Proposition 3.1.** For \(k = 0, ..., n - 1,\)

\[(3.3) \quad s_{(k+1,1^{n-k-1})}[X/(1 - q)] = \sum_{r=k+1}^{n} T_{k+1,r} \frac{E_{n,r}(X)}{(q; q)_r}.\]

**Proof.** Using the Cauchy \(q\)-binomial theorem, we see that the coefficient of \((-z)^k\) on the right hand side of (3.1) is

\[(3.4) \quad q^k \sum_{i=0}^{n-k} \left[ \begin{array}{c} k + i \\ k \end{array} \right] \frac{E_{n,k+i}}{(q; q)_{k+i}}.\]

On the other hand, by the identities (2.16) and (2.18),

\[
\begin{align*}
    e_n[X^{1-z}/(1-q)] & = \sum_\lambda s_\lambda[X^{1-z}/(1-q)] s_\lambda[1-z] \\
    & = \sum_{\lambda'=(n-r,1^r)} s_\lambda[X^{1-q}/(1-q)] (-z)^\lambda (1-z) \\
    & = \sum_{r=0}^{n-1} s_{(r+1,1^{n-r-1})}[X^{1-q}/(1-q)] (-z)^\lambda (1-z),
\end{align*}
\]
which is equal to
\[ s_{1n}\left[\frac{X}{1-q}\right] + (-z)(s_{1n}\left[\frac{X}{1-q}\right] + s_{2,1n-2}\left[\frac{X}{1-q}\right]) + \cdots + (-z)^n s_{n}\left[\frac{X}{1-q}\right]. \]

Comparing the coefficient of \((-z)^k\) gives, for \(k \geq 1\),
\[ q^{(k)} \sum_{i=0}^{n-k} \begin{bmatrix} k \end{bmatrix}_j E_{n,k+i}(q; q) = s_{k,1n-k}\left[\frac{X}{1-q}\right] + s_{k+1,1n-k-1}\left[\frac{X}{1-q}\right], \]
and
\[ \sum_{i=1}^{n} \frac{E_{n,i}(q; q)}{(q; q)_i} = s_{1n}\left[\frac{X}{1-q}\right]. \]

We take the alternating sums of the equations (3.5) and (3.6) to get
\[ s_{k+1,1n-k-1}\left[\frac{X}{1-q}\right] = (-1)^k \left( \sum_{j=1}^{n} E_{n,j}(q; q)_j \right) + \sum_{i=1}^{k} (-1)^{k+i} q^{(i)} \sum_{j=0}^{n-i} \begin{bmatrix} i+j \end{bmatrix}_j E_{n,i+j}(q; q)_{i+j}. \]

By collecting \(E_{n,k}(X)\)’s, and using (2.19) we obtain
\[ s_{(k+1,1n-k-1)}\left[\frac{X}{1-q}\right] = \sum_{r=k+1}^{n} T_{k+1,r} \frac{E_{n,r}(X)}{(q; q)_r}. \]

Let \(S\) and \(E\) be the matrices
\[
S = \begin{pmatrix}
s_{1n}\left[\frac{X}{1-q}\right] \\
s_{2,1n-1}\left[\frac{X}{1-q}\right] \\
\vdots \\
s_{n}\left[\frac{X}{1-q}\right]
\end{pmatrix}
\]
and
\[
E = \begin{pmatrix}
\frac{E_{n,1}(q; q)}{(q; q)_1} \\
\frac{E_{n,2}(q; q)_2}{(q; q)_2} \\
\vdots \\
\frac{E_{n,n}(q; q)_n}{(q; q)_n}
\end{pmatrix},
\]
respectively, and let \(T\) be the transition matrix from \(E\) to \(S\), so that \(S = TE\). Then, \(T\) is an upper triangular matrix with the \(k+1, r\)'th entry
\[ T_{k+1,r} = (-1)^k \sum_{i=0}^{k} (-1)^i q^{(i)} \begin{bmatrix} i \end{bmatrix}_r. \]
For example, when $n = 5$, 

$$T = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & q & (q + 1)q & (q^2 + q + 1)q & (q^3 + q^2 + q + 1)q \\
0 & 0 & q^3 & q^3(q^2 + q + 1) & q^3(q^3 + q^2 + 2q^2 + q + 1) \\
0 & 0 & 0 & q^6 & q^6(q^3 + q^2 + q + 1) \\
0 & 0 & 0 & 0 & q^{10}
\end{pmatrix}.$$ 

Then, 

$$T^{-1} = \begin{pmatrix}
1 & -q^{-1} & q^{-2} & -q^{-3} & q^{-4} \\
0 & q^{-1} & -q^{-1}q & q^2+q+1 & -q^3+q^2+q+1 \\
0 & 0 & q^{-3} & -q^2+q+1 & q^4+q^4+2q^2+q+1 \\
0 & 0 & 0 & q^{-6} & -q^3+q^2+q+1 \\
0 & 0 & 0 & 0 & q^{-10}
\end{pmatrix}.$$ 

**Proposition 3.2.** $T^{-1}$ is (necessarily) upper triangular, and its $k + 1, r$'th entry is equal to 

$$(T^{-1})_{k+1,r} = (-1)^{r-k}q^{-r(k+1)}T_{k+1,r}.$$ 

**Proof.** Let $L$ be the upper triangular matrix with the $k + 1, r$'th entry 

$$L_{k+1,r} = (-1)^{r-k}q^{-r(k+1)}T_{k+1,r} \quad \text{for } r > k.$$ 

Clearly, $TL$ is an upper triangular matrix, and the $i + 1, j$'th entry of $TL$ is 

$$(TL)_{i+1,j} = \sum_{k=1}^{n} T_{i+1,k} L_{k,j}. $$

It is straightforward to check that $(TL)_{i+1,i+1} = 1$. We use induction on $j$ to prove that for all $i + 1 < j$, $(TL)_{i+1,j} = 0$. So, we assume that for all $i + 1 < j$, $(TL)_{i+1,j} = 0$, and we are going prove that for all $i+1 < j+1$, $(TL)_{i+1,j+1} = 0$. 

First of all, using the $q$-binomial identity 

$${r \choose m} = \left[ {r-1 \choose m} q^r q^{-m}, \text{ for } m \geq 0, \right]$$

it is easy to show that 

$$(T_{i+1,k}) = T_{i+1,k-1} + q^i T_{i,k-1}.$$ 

It follows that 

$$(L_{k+1,j+1}) = -q^{-(k+1)}L_{k+1,j} + q^{-k}L_{k,j-1}.$$ 

Therefore,
\[
\sum_{k=i+1}^{j+1} T_{i+1,k} L_{k,j+1} = \sum_{k=i+1}^{j+1} T_{i+1,k} (-q^{-(k+1)} L_{k,j} + q^{-k} L_{k-1,j-1})
\]
\[
= \sum_{k=i+1}^{j+1} -q^{-k} T_{i+1,k} L_{k,j} + \sum_{k=i+1}^{j+1} q^{-(k-1)} T_{i+1,k} L_{k-1,j}.
\]

Using (3.10) in the last summation, we have
\[
\sum_{k=i+1}^{j+1} T_{i+1,k} L_{k,j+1} = \sum_{k=i+1}^{j+1} -q^{-k} T_{i+1,k} L_{k,j} + \sum_{k=i+1}^{j+1} q^{-k} T_{i+1,k} L_{k,j} + \sum_{k=i+1}^{j+1} q T_{i+1,k-1} L_{k-1,j}.
\]

After rearranging the indices, and using the induction hypotheses, the right hand side of the equation simplifies to 0. Therefore, the proof is complete. □

**Corollary 3.3.** Let \( A \subseteq \Lambda^n_{\mathcal{Q}(q)}(X) \) be the \( n \)-dimensional subspace generated by the set \( \{ E_{n,k}(X) \}_{k=1}^{n} \), and let \( B \subseteq \Lambda^n_{\mathcal{Q}(q)}(X) \) be the \( n \)-dimensional subspace generated by \( \{ s_{k,1^{n-k}}[X/(1-q)] \}_{k=1}^{n} \). Then, \( A = B \).

**Proof.** It is clear by Proposition 3.2 that \( A = B \). The dimension claim follows from Proposition 4.1 below. □

The expression \((-1)^n p_n = \sum_{k=0}^{n-1} (-1)^k s_{k+1,1^{n-k-1}}\) is the bridge between Schur functions of hook type with the power sum symmetric functions. By the linearity of plethysm we have
\[
(-1)^n p_n [X/(1-q^n)] = (-1)^n p_n [X/(1-q)] = \sum_{k=0}^{n-1} (-1)^k s_{k+1,1^{n-k-1}} [X/(1-q)],
\]
and therefore
\[
(3.12) \quad (-1)^n p_n = (1 - q^n) \sum_{k=0}^{n-1} (-1)^k s_{k+1,1^{n-k-1}} [X/(1 - q)].
\]

**Corollary 3.4.** For all \( n \geq 1 \),
\[
(3.13) \quad (-1)^n p_n = \sum_{r=1}^{n} \frac{1 - q^n}{1 - q^r} E_{n,r}.
\]
Proof. By Proposition 3.1 and (3.12) we get

\[(3.14) \quad (-1)^n p_n = (1 - q^n)^{n-1} \sum_{k=0}^{n-1} \sum_{r=k+1}^{n} (-1)^k T_{k+1,r} \frac{E_{n,r}(X)}{(q; q)_r}. \]

By rearranging the summations and using the Cauchy’s $q$-binomial theorem once more, we finish the proof. \(\square\)

4. Hall-Littlewood expansion.

**Proposition 4.1.** For $k = 1, \ldots, n$,

\[
\frac{E_{n,k}(X)}{(q; q)_k} = \sum_{\mu \in P_{nr}(n,k)} \frac{\bar{H}_{\mu'}(X; q)}{\bar{h}_{\mu}(q,0)\bar{h}'_{\mu}(q,0)} = \sum_{\mu \in P_{nr}(n,k)} \frac{(-q)^n q^{2\mu'}(\mu')}{\prod_{s \in \mu, \ell_s(\mu) = 0} (-q^{a_{\mu,s}(s)+1}) \prod_{s \in \mu, \ell_s(\mu) = 0} (1 - q^{-a_{\mu,s}(s)-1})}.
\]

Proof. Let $Y = (1 - t)(1 - z)$. Then, by the Cauchy identity (2.17), we have

\[(4.1) \quad \sum_{k=1}^{n} z^k q_k \frac{E_{n,k}(X)}{(q; q)_k} = \sum_{\mu \in \nu} \frac{\bar{H}_{\mu}(X; q)}{\bar{h}_{\mu}(q,t)\bar{h}'_{\mu}(q,t)}. \]

The left hand side of the equation (4.1) is independent of the variable $t$. Since $\bar{h}_{\mu}(q,0) \neq 0$, and since $\bar{h}'_{\mu}(q,0) \neq 0$, we are allowed to make the substitution $t = 0$ on both sides of the equation.

Note that

\[
\bar{h}_{\mu}(q,0)\bar{h}'_{\mu}(q,0) = \prod_{s \in \mu} q^{a_{\mu,s}(s)} \prod_{s \in \mu, \ell_s(\mu) = 0} (-q^{a_{\mu,s}(s)+1}) \prod_{s \in \mu, \ell_s(\mu) = 0} (1 - q^{-a_{\mu,s}(s)-1}).
\]

The equality (4.2) follows from (2.1).

Using the Schur expansion $\bar{H}_{\mu}(X; q, t) = \sum_{\lambda} \bar{K}_{\lambda \mu}(q, t)s_{\lambda}$, we see that the plethystic substitution $X \rightarrow (1 - z)$, followed by the evaluation at $t = 0$ is the same as the evaluation $\bar{H}_{\mu}(X, q, 0)$ at $t = 0$, followed by the plethystic substitution $X \rightarrow (1 - z)$. Also, by Corollary 3.5.20 of [6], we know that

\[
\bar{H}_{\mu}[1 - z; q, t] = \Omega \bar{B}_{\mu} \begin{pmatrix} t \end{pmatrix} - z^\mu, \text{ where } B_{\mu} = \sum_{i\geq 1} t^{i-1} \frac{1 - q^{a_{\mu,i}}}{1 - q}.
\]
Therefore,
\begin{align}
\tilde{H}_{\mu}[1 - z; q, 0] &= \Omega[-zB_{\mu}]_{t=0} \\
&= \Omega[-z(1 + q + \cdots + q^{\mu_1-1})] \\
&= \prod_{i=0}^{\mu_1-1} (1 - zq^i) \\
&= (z; q)_{\mu_1}.
\end{align}
(4.3)

It follows from (2.11) and (2.12) that
\begin{equation}
\tilde{H}_{\mu}(X; q, 0) = \tilde{H}_{\mu'}(X; q).
\end{equation}
(4.7)

By combining (4.1), (4.2), (4.6) and (4.7), we get
\begin{equation}
\sum_{n} (z; q)_{k} E_{n,k}(X) = \frac{1}{(q;q)_n} \sum_{\mu \vdash n} (z; q)_{\mu_1} \prod_{s \in \mu, j \mu(s) = 0} (1 - q^{-a_{\mu}(s)-1}).
\end{equation}
(4.8)

By comparing the coefficient of \((z; q)_{k}\) in (4.8), we find that
\begin{equation}
E_{n,k}(X) = \sum_{\mu \in \mathcal{P}_{\text{arr}(n,k)}} \frac{\tilde{H}_{\mu'}(X; q)}{(q; q)_{k}} \prod_{s \in \mu, j \mu(s) = 0} (1 - q^{-a_{\mu}(s)-1}).
\end{equation}
(4.9)

Hence, the proof is complete.

Lemma 4.2. Let \(\lambda \vdash n\) be a partition of \(n\). Then,
\begin{equation}
\frac{X}{(1 - q)(1 - t)} = \sum_{\mu \vdash n} \frac{\tilde{K}_{\lambda \mu'}(q, t)\tilde{H}_{\mu}(X; q,t)}{h_{\mu}(q, t)h'_{\mu}(q, t)}.
\end{equation}
(4.10)

Proof. This follows from Theorem 1.3 of [3].

Corollary 4.3. Let \(\lambda \vdash n\) be a partition. Then,
\begin{equation}
\frac{X}{1 - q} = \sum_{\mu} \frac{\tilde{K}_{\lambda \mu'}(q)\tilde{H}_{\mu'}(X; q)}{h_{\mu}(q, 0)h'_{\mu}(q, 0)} = \sum_{\mu} \frac{\tilde{K}_{\lambda \mu'}(q)\tilde{H}_{\mu'}(X; q)}{(-q)^n q^{2n(\mu') \prod_{s \in \mu, j \mu(s) = 0} (1 - q^{-a_{\mu}(s)-1})}}.
\end{equation}
(4.11)

Proof. It follows from (2.12) and (2.13) that \(\tilde{K}_{\lambda \mu'}(q, 0) = \tilde{K}_{\lambda \mu'}(0, q) = \tilde{K}_{\lambda \mu'}(q)\). Since,
\begin{align*}
\tilde{h}_{\mu}(q, 0)\tilde{h}'_{\mu}(q, 0) &= (-q)^n q^{2n(\mu')} \prod_{s \in \mu, j \mu(s) = 0} (1 - q^{-a_{\mu}(s)-1}),
\end{align*}
and \(\tilde{H}_{\mu}(X; q, 0) = \tilde{H}_{\mu'}(X; q)\), the proof follows from Lemma 4.2.
Theorem 4.4. Let $1 \leq k \leq r \leq n$, and let $\mu \in Par(n, r)$. Then,

$$T_{k,r} = \widetilde{K}_{(n-k+1, 1^{k-1})}^{\mu}(q).$$

Proof. Recall that

$$s_{k+1,1^{n-k-1}} \left[ X \frac{1}{1-q} \right] = \sum_{r=k+1}^{n} T_{k+1,r} \frac{E_{n,r}(X)}{(q;q)_r},$$

where

$$T_{k+1,r} = (-1)^k \sum_{i=0}^{k} (-1)^i q^{\binom{i}{2}} \left[ \begin{array}{c} r \\ i \end{array} \right].$$

Therefore, by Corollary 4.3 and Proposition 4.1 we have

$$\sum_{\mu} \frac{\widetilde{K}_{(n-k+1, 1^{k-1})}^{\mu}(q) \widetilde{H}_{\mu}(X; q)}{\overline{h}_{\mu}(q, 0) \overline{h}'_{\mu}(q, 0)} = s_{k,1^{n-k}} \left[ X \frac{1}{1-q} \right]$$

$$= \sum_{r=k}^{n} T_{k,r} \sum_{\mu \in Par(n, r)} \frac{\widetilde{H}_{\mu}(X; q)}{\overline{h}_{\mu}(q, 0) \overline{h}'_{\mu}(q, 0)}$$

$$= \sum_{\mu \in \mathcal{U}_{n-k}^r Par(n, r)} \frac{T_{k,r} \widetilde{H}_{\mu}(X; q)}{\overline{h}_{\mu}(q, 0) \overline{h}'_{\mu}(q, 0)}.$$

The theorem follows from comparison of the coefficients of $\widetilde{H}_{\mu}(X; q)$. □

REFERENCES


