Special Embeddings of Symmetric Varieties and their Borel orbits

Mahir Bilen Can[1], Roger Howe[2], and Lex Renner[3]

[1]Tulane University, New Orleans; mcan@tulane.edu
[2]Yale University, New Haven; howe@yale.edu
[3]University of Western Ontario; lex@uwo.edu

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Abstract

By using the theory of linear algebraic monoids, we introduce a new class of spherical embeddings of symmetric varieties. Along the way we determine when an antiinvolution on a semisimple linear algebraic group G extends to an antiinvolution on a J-irreducible monoid whose group of invertible elements is G. Furthermore, extending the work of Springer on involutions, we describe the parametrizing sets of Borel orbits in our embeddings.

Key words: Borel orbits, Reductive monoids, symmetric spaces

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1 Introduction

Let G be a connected linear algebraic group defined over a field k. For simplicity we assume that k is algebraically closed and of characteristic zero. A closed subgroup H of G is called *spherical*, if there exists a Borel subgroup $B \subseteq G$, that is to say a maximally closed connected solvable subgroup, such that the group generated by the elements $b \cdot h$, where $b \in B$, $h \in H$ forms an open subset in G. Equivalently, the coset space G/H, on which G-acts algebraically on the left has a dense B-orbit. More generally, a normal G-variety X is called *spherical* (or, *G-spherical*), if the subgroup B has a dense orbit. By the works of Brion [1], Popov [19], and Vinberg [35], we know that X is spherical if and only if X is comprised of finitely many B-orbits. Spherical varieties are abundant in mathematics. Among the familiar examples to algebraic geometers are the torus embeddings, as well as homogenous spaces.

A special but very important class of spherical varieties arises from a study of automorphisms on linear group. In particular, automorphisms of order two, namely *involutions*, are pivotal for theory of real forms. It is shown by Matsuki [16] (when $k = \mathbb{C}$, the field of complex numbers) and by Springer [32] (for general k) that the fixed subgroup $G^\theta = \{g \in G : \theta(g) = g\}$ of an involution on G is a spherical subgroup. Coset spaces of the form G/G^θ, where θ is an involution, are called *symmetric varieties*. Over the field of real numbers, such manifolds are known as "Riemannian symmetric spaces." The isomorphism types of compact Riemennian spaces are described by Cartan in [4], and more uniformly by Huang and Leung in [12]. Other prominent examples of spherical varieties appear in the modern theory of group compactifications: X is said to be an *embedding* of G/H, if it has an open subset that is isomorphic to G/H. Clearly, if H is a spherical subgroup, then X is a spherical variety. When G is a product group of the form $K\times K$, and H is the diagonal copy of K in G, then G/H is isomorphic to K as a $K\times K$-variety. In [29], Rittatore shows that an affine $K\times K$-variety X is an embedding of the group K if and only if X is a linear algebraic monoid, that is to say, a closed, unital subsemigroup of the monoid of $n\times n$ matrices for some n. Moreover, in this case, K is the group of invertible elements of the monoid X.

Roughly speaking our main results are; a construction of a new class of spherical embeddings of symmetric spaces using monoids, and a description of the sets parameterizing Borel orbits therein. Before getting into the details of our results, we present an observation, along the way of which we introduce some concepts that we need in the sequel:

Suppose G is semisimple and of adjoint type, and $\theta : G \to G$ is an involutory automorphism. Let $\widetilde{H}$ denote the normalizer of the fixed subgroup $H = G^\theta$ in G. According to [6], there exists a unique normal G-variety X such that

i) there exists an open G-orbit $X_0 \subset X$ that is isomorphic to $G/\widetilde{H}$,

ii) the complement $X - X_0$ is a finite union $\cup_{\alpha\in S} X_\alpha$ of smooth, G-stable boundary divisors with normal crossings,

iii) any G-orbit closure is of the form $\cap_{\alpha\in I} X_\alpha$ for some subset $I \subset S$, where S is as in ii).

The variety X is called the *wonderful compactification* of $G/\widetilde{H}$. In the case when G is the doubled group $K\times K$, and the involution θ is $\theta((h,k)) = (k,h)$ for $(h,k) \in K\times K$, the wonderful compactification of $G/\widetilde{H}$ is closely related to a particular linear algebraic monoid. To explain, we recall some standard algebraic monoid definitions. A *reductive* monoid is, by definition, a linear algebraic monoid with a reductive group of invertible elements. A *J-irreducible* monoid is a reductive monoid with a unique closed $G\times G$-orbit that has at least one nonzero element. Such monoids are abundant in representation theory. Let (ρ, V) be a finite dimensional irreducible representation of a semisimple group G_0. The image of ρ_0 lies in the affine space of linear operators $\mathrm{End}(V)$. By abusing notation, denote by k^* the group of nonzero scaling operators on V and let G denote the product subgroup $k^*\cdot\rho(G_0)$ of $\mathrm{GL}(V)$. The Zariski closure of G in $\mathrm{End}(V)$ is a J-irreducible monoid. To obtain a wonderful compactification of G_0, a semisimple group of adjoint type, we start with a representation V with its highest weight in general position such that $\dim(V\otimes V)^{G_0} = 1$. Normalizing the projectivization of the associated J-irreducible monoid gives us the wonderful compactification of G_0. See [25].

The special relationship between the wonderful compactification of G and the monoid described at the end of the previous paragraph is not accidental. Indeed, the juxtaposition of these two varieties is an irreducible representation whose highest weight is in general position. To exploit this point of view in a more general setting, we fist embed the symmetric space G/H in a J-irreducible monoid by using a well-known observation due to Vust: Let $\tau : G \to G$ be the morphism defined by

$$\tau(g) = g\theta(g)^{-1} \tag{1.1}$$

It is shown by Vust in [36], and later simplified by Richardson in [27] that the image of τ in G, denoted by $P = \{\tau(g) : \ g \in G\}$, is isomorphic to G/H. Observe that in the case when the group G is $K \times K$ with the involution $\theta(h,k) = (k,h)$ for all $k,h \in K$, P is equal to the whole variety K.

Next we consider a J-irreducible monoid M whose group of invertible elements of G. Clearly, the closure of P in M gives us an embedding of G/H. At this point it is desirable to determine if the defining involution of P has some sort of extension to M, or not. Paraphrasing this as a question, we ask:

"For which monoids M do we have an involution to work with?"

Our first main result answers this question in the absence of invertibility.

Theorem 1. Let θ be an involutory automorphism of G, and let H be the fixed subgroup $H = G^\theta$. Let (V,ρ) be an irreducible representation of G with highest weight μ, and let M denote the associated J-irreducible monoid. If the induced map θ^* on weights satisfies $\theta^*(\mu) = -\mu$, then there exists a unique morphism $\theta_{an} : M \to M$ such that

1. $\theta_{an}(xy) = \theta_{an}(y)\theta_{an}(x)$ for all $x, y \in M$,
2. θ_{an}^2 is the identity map on M,
3. $\theta_{an}(g) = \theta(g)^{-1}$ for all $g \in G$.

As in the case of a homogenous space, orbits of a Borel subgroup dictate the geometry of a symmetric space. In [32], Springer determines the structure of these orbits. The inclusion posets of orbit closures are investigated by Matsuki in [17] (for $\mathbb{C}$), and by Richardson and Springer in [28] (for arbitrary k with characteristic zero). These results are sharpened and extended over the fields of arbitrary characteristic (not two) by Helminck and Wang in [10], and by Helminck in [9]. Much of what we describe next provides us with a new ground of applications of these results. Before stating our next theorem in this direction, let us briefly summarize what is known for B-orbits in an arbitrary symmetric space.

Let $T \subseteq B$ be a θ-stable maximal torus of a Borel subgroup B of G. The *Weyl group* W of G is the finite group $N_G(T)/T$, where $N_G(T)$ is the normalizer of T in G. Let $\mathcal{V}$ denote the set of all $g \in G$ such that $\tau(g) \in N_G(T)$, where τ is as in (1.1). It is easy to verify that $\mathcal{V} \subset G$ is closed under the following action of $T \times H$ on G;

$$(t,h) \cdot g = tgh^{-1} \ \text{ for } t \in T, h \in H, \text{ and } g \in G.$$

Let V denote the set of $T \times H$-orbits in $\mathcal{V}$. For $v \in V$, let $v(x) \in \mathcal{V}$ denote a representative of the orbit v. Then it is known that the inclusion $\mathcal{V} \hookrightarrow G$ induces a bijection from V onto the set of $B \times H$-orbits in G. In particular, G is the disjoint union of the double cosets $Bx(v)H$, $v \in V$. See [9].

We continue with the assumption that M is a reductive monoid with an antiinvolution $\theta_{an} : M \to M$. The composition of θ_{an} with the inverse map $\iota(g) = g^{-1}$, $g \in G$, gives us an involution θ on the unit group G. Let H denote the fixed subgroup G^{θ}, and let $P \subset G$ be the closed subvariety of G that is isomorphic to G/H. We define the *twisted action* of G on M by

$$g * m = gm\theta_{an}(g) = gm\theta(g)^{-1},\ g \in G, m \in M. \tag{1.2}$$

Clearly, P is stable under (1.2). In fact, $P = G * 1_G$, where 1_G is the identity element of G. Since P is open in its closure $\overline{P}$ in M, $\overline{P}$ is closed under (1.2). In particular, $\overline{P}$ is a spherical variety. An important auxiliary variety for the study of symmetric spaces is defined by

$$Q = \{g \in G :\ \theta(g) = g^{-1}\}.$$

Clearly, Q is a closed subvariety of G, and it is stable under the twisted action. In fact, Q is the fixed subvariety of the antiinvolution θ_{an} on G. We know from [32] that it is a spherical G-variety. Indeed, for $g \in Q$, applying θ to $h * g = hg\theta(h)^{-1}$, $h \in G$, we see that $\theta(h * g) = \theta(h)g^{-1}h^{-1} = (hg\theta(h))^{-1} = (h * g)^{-1}$. Also, since $\theta(g\theta(g)^{-1}) = \theta(g)g^{-1} = (g\theta(g^{-1}))$, we see that $P \subseteq Q$.

Following Springer's development that uses Q to study P, we consider an analog of Q in the monoid setting. Define

$$M_Q := \{m \in M :\ \theta_{an}(m) = m\}. \tag{1.3}$$

Since θ_{an} is a morphism, and M is separated, M_Q is a closed subset in M. It follows from the inclusions $P \subseteq Q \subseteq M_Q$ that $\overline{P} \subseteq \overline{Q} \subseteq M_Q$.

G acts on the sets $P, Q, \overline{P}, \overline{Q}$, and on M_Q by the same formula $g * m = gm\theta_{an}(g)$. Our goal is give a satisfactory description of the $B*$-orbits in these varieties. Note that, $B*$-orbits in G are in one-to-one correspondence with the B-orbits in G/H.

Theorem 2. Let $T \subseteq G$ be a θ-stable maximal torus in a θ-stable Borel subgroup $B \subset G$, and let $\overline{N}$ denote the Zariski closure in M of the normalizer $N_G(T)$. The following sets are finite and they are in bijection with each other:

1. $B*$-orbits in $\overline{Q}$ (respectively, $B*$-orbits in $\overline{P}$),
2. $T \times H$-orbits in $\tau^{-1}(\overline{N} \cap \overline{Q})$ (respectively, $T \times H$-orbits in $\tau^{-1}(\overline{N} \cap \overline{P})$).

Before closing our introduction by presenting the contents of our paper, we comment on consequences of Theorem 2. The *Renner monoid*, $R = \overline{N}/T$ generalizes the notion of Weyl group from linear algebraic groups to reductive monoids [24]. Its elements are in bijection with the $B \times B$-orbits in M, and its system of idempotents encodes crucial information on the structure of a suitable torus embedding. Its inverse semigroup structure encapsulates many interesting combinatorial properties. See [15]. The partial ordering on R that is induced from by inclusion relations among the closures of $B \times B$-orbits plays an important role for the geometry of M and for the related algebras. See [30], [21], [22], and [18]. Theorem 2 opens a way for many related enumerative and combinatorial questions. A direct consequence of Theorem 2 is that a certain subset of R parametrizes the $B*$-orbits in $\overline{P}$. What is the size of this subset of R? Of course, the same question makes sense for the $B*$-orbits in $\overline{Q}$.

We structured our paper as follows. In Section 2 we recall some well-known facts about representation theory of semisimple linear groups, in particular the theory of weights. Also, we review basics of the theory of linear algebraic monoids, including important subclass of J-irreducible monoids, and the convex geometry of their maximal tori. In Section 3, after recalling the theory of root system related to symmetric spaces, we prove our first result, namely Theorem 1. The definition of special compactifications, generalizing the notion of wonderful compactification is given in Section 4 to motivate and to provide the background for what comes next. In Section 5 we obtain the characterization of the set of Borel orbits in embeddings that are defined earlier. In particular we prove Theorem 2 as a corollary of the related result which describes the set of Borel orbits of M_Q. We close our paper with some final remarks in Section 6.

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2 Preliminaries

In this section we introduce our basic notation and background for algebraic groups and monoids. For representation theory related notation we follow [13], and [14], and for monoids we follow [26], and [20].

We denote by GL_n the general linear group of invertible $n \times n$ matrices, and denote by Mat_n the monoid of $n \times n$ matrices. A linear algebraic group is a closed subgroup of GL_n. More generally, a linear algebraic monoid is a closed submonoid of Mat_n. We always assume that a linear algebraic group is connected.

The Lie algebra of a linear algebraic group G is denoted by $\mathrm{Lie}(G)$, and the *adjoint representation* $(\rho, V) = (\mathrm{Ad}, \mathrm{Lie}(G))$ of G is defined by the action

$$g \cdot A = gAg^{-1} \quad \text{for} \quad g \in G, A \in \mathrm{Lie}(G).$$

G is called simple if it has no non-trivial, normal, connected subgroup. Equivalently, the Lie algebra of G has no non-trivial abelian ideals. In this case, $\mathrm{Lie}(G)$ is said to be a simple Lie algebra. G is called ssemisimple, if its Lie algebra is *semisimple*, that is to say $\mathrm{Lie}(G)$ is a sum of simple Lie algebras. More generally, G is called reductive if its Lie algebra is the sum of an abelian and a semisimple Lie algebra. Throughout the paper, we consider reductive groups, only.

Our final general assumption on groups and monoids is that all representations are rational and finite dimensional.

2.1 Weights

If $T \subseteq G$ is a maximal torus, then a *weight* of T on a representation (π, V) of G is a character $\lambda : T \to k^*$ such that the subspace $V_\lambda = \{v \in V : \ \pi(t) \cdot v = \lambda(t)v \text{ for all } t \in T\}$ is nonempty. We denote by $X(V)$ the (finite) set of weights of a given representation (π, V). Collection of all weights (from all representations) of T forms a free abelian group, denoted by $X(T)$. The real vector space $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ is denoted by E. For a Borel subgroup B containing T, we have the following additional notation:

$$\begin{aligned} \varPhi \subset X(T) & \text{ the set of weights of the adjoint representation,} \\ \varDelta \subset \varPhi & \text{ the set of simple roots determined by } (B,T), \\ \varPhi^+ \subset \varPhi & \text{ the subset of positive roots associated with } \varDelta. \end{aligned}$$

Recall that the *Weyl group* of a semisimple group (more generally, of a reductive group) G is the finite quotient group $W = N_G(T)/T$, where $N_G(T)$ is the normalizer of a maximal torus T in G. There is a faithful representation of W on E by which W is recognized as a "reflection group." In other words, there exists a W-invariant inner product $(\, ,) : \ E \times E \to \mathbb{R}$ and the group generated by the orthogonal transformations

$$\sigma_\alpha(\mu) = \mu - \frac{2(\mu, \alpha)}{(\alpha, \alpha)} \alpha, \ \ \mu \in E, \alpha \in \varPhi \tag{2.1}$$

is isomorphic to W.

Let $\langle \, , \rangle : E \times E \to \mathbb{R}$ denote the pairing defined by $\langle \lambda, \mu \rangle = 2(\lambda, \mu)/(\mu, \mu)$, $\mu, \lambda \in E$. An *abstract weight* in E is a vector $\lambda \in E$ such that $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ for all $\alpha \in \varPhi$. In particular $X(T)$ consists of abstract weights. The set of all abstract weights in E forms a lattice (of full rank) denoted by $\varLambda$, and G is called *simply-connected*, if $\varLambda = X(T)$. The sublattice of $X(T)$ generated by the set of all roots is called the *root lattice*, denoted by $\varLambda_r$. G is called of *adjoint type*, if $X(T) = \varLambda_r$.

If $\varDelta$ is $\{\alpha_1, \dots, \alpha_\ell\}$, then the *fundamental dominant weights* ω_i's are defined by the requirements $\langle \omega_i, \alpha_j \rangle = \delta_{i,j}$ for $i, j = 1, \dots, \ell$. They form a basis for the lattice $\varLambda$, as well as for the vector space E. A vector $\lambda \in E$ is called *dominant* if all c_i's in its expansion $\lambda = \sum_{i=1}^{\ell} c_i \omega_i$ are non-negative integers. If for all $i = 1, \dots, \ell$ the integer c_i is positive, then λ is called a *regular (dominant) weight*. By its representation defined in (2.1), W acts on $\varLambda$, furthermore, each $\lambda \in \varLambda$ is W-conjugate to a unique dominant weight.

There is an important partial ordering $\leqslant$ on E defined by $\mu \leqslant \lambda$ if $\lambda - \mu$ is a non-negative integer combination of positive roots. W permutes the weights $X(V)$ of (π, V), and there is a unique weight $\lambda \in X(V)$ such that $\mu \leqslant \lambda$ for all $\mu \in X(V)$. In this case, λ is called the *highest weight* of the representation (π, V).

2.2 Reductive monoids

The purpose of this section is to introduce the notation of the reductive monoids. We focus mainly on the orbit structure of these monoids. In addition to Renner and Putcha's books mentioned above, we recommend more recent semi-expository article by Brion [2]. For an introduction from a combinatorial point of view, with many explicit examples, we suggest the survey by Solomon [31]. For Renner monoids, we recommend Cao, Li and Li's survey article [15].

Let M be linear algebraic monoid. The set $G = G(M)$ of invertible elements of M is an algebraic group. If G is a reductive group and M is an irreducible variety, then M is called a *reductive monoid*. In a reductive monoid, the data of the Weyl group W of (G, T) and the set $E(\overline{T})$ of idempotents of the embedding $\overline{T} \hookrightarrow M$ combine to become a finite inverse semigroup $R = \overline{N_G(T)}/T \cong W \cdot E(\overline{T})$ with unit group W and idempotent set $E(R) = E(\overline{T})$. Recall that the Bruhat-Chevalley order W is defined by

$$x \leq y \quad \text{if and only if} \quad BxB \subseteq \overline{ByB}.$$

Similarly, on the Renner monoid R of M, the Bruhat-Chevalley order is defined by

$$\sigma \leq \tau \quad \text{if and only if} \quad B\sigma B \subseteq \overline{B\tau B}. \tag{2.2}$$

The induced poset structure on W, which is induced from R is the same as the original Bruhat poset structure on W.

Let $E(\overline{T})$ denote the set of idempotent elements in the (Zariski) closure of $T \subseteq G$ in the monoid M. Similarly, let us denote by $E(M)$ the set of idempotents in the monoid M. Plainly we have $E(\overline{T}) \subseteq E(M)$. There is a canonical partial order $\leq$ on $E(M)$ (hence on $E(\overline{T})$) defined by

$$e \leq f \quad \text{if and only if} \quad ef = e = fe. \tag{2.3}$$

Notice that $E(\overline{T})$ is invariant under the conjugation action of the Weyl group W. A subset $\Lambda \subseteq E(\overline{T})$ is called a *cross-section lattice* (or, a *Putcha lattice*) if Λ is a set of representatives for the W-orbits on $E(\overline{T})$ and the bijection $\Lambda \to G\backslash M/G$ defined by $e \mapsto GeG$ is order preserving. It turns out that we can write $\Lambda = \Lambda(B) = \{e \in E(\overline{T}) : \ Be = eBe\}$ for some unique Borel subgroup B containing T. The partial order given by (2.3) on $E(\overline{T})$ (hence on Λ) agrees with the Bruhat-Chevalley order (2.2) on the Renner monoid.

The decomposition $M = \bigsqcup_{e\in\Lambda} GeG$ into $G \times G$ orbits has a counterpart on the Renner monoid. Namely, the finite monoid R can be written as a disjoint union

$$R = \bigsqcup_{e\in\Lambda} WeW$$

of $W \times W$ orbits, parametrized by the cross section lattice.

$E(\overline{T})$ is a relatively complemented lattice, anti-isomorphic to a face lattice of a convex polytope. For $\overline{T}$ contained in a J-irreducible monoid, the associated polytope is described explicitly in Section 2.3. Let Λ be a cross section lattice in $E(\overline{T})$. The Weyl group of T (relative to $B = C_G^r(\Lambda)$) acts on $E(\overline{T})$, and furthermore

$$E(\overline{T}) = \bigsqcup_{w \in W} w\Lambda w^{-1}.$$

2.3 J-irreducible monoids and their cones

Let G_0 be a simple linear algebraic group and let T_0 be a maximal torus in G_0. If (ρ_0, V) is an n-dimensional irreducible representation of G_0 such that $\dim \ker(\rho_0) = 0$, then the group $k^*\rho_0(G_0)$, which we denote by G is reductive. Up to isomorphism T_0 and $\rho_0(T_0)$ differ by a finite set of central elements, so, when there is no danger of confusion, by abusing notation, we denote the image $\rho_0(T_0)$ by T_0, also. Let $T \subseteq G$ be a maximal torus containing T_0, and let $\mathbf{T} \subset \mathrm{GL}(V)$ denote an n-dimensional maximal torus containing T. Accordingly, we have a nested sequence of Euclidean spaces:

$$E_0 = X(T_0) \otimes_{\mathbb{Z}} \mathbb{R} \subset E = X(T) \otimes_{\mathbb{Z}} \mathbb{R} \subset \mathbf{E} = X(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Note that $\dim E = \dim E_0 + 1$.

If ε_i, $i = 1, \dots, n$ denote the standard coordinate functions on $\mathbf{T}$, then $\{\varepsilon_1, \dots, \varepsilon_n\}$ is a basis for $\mathbf{E}$, and E is spanned by the restrictions $\varepsilon_i|_T$, $i = 1, \dots, n$. Let $\chi \in X(T)$ denote the restriction of the character whose n-th power is the determinant on $\mathrm{GL}(V)$. Stated differently in additive notation, χ is the restriction to T of the rational character $\frac{1}{n}(\varepsilon_1 + \cdots + \varepsilon_n)$. We denote $\varepsilon_i|_T$ by χ_i and set

$$\widetilde{\chi}_i := \chi_i - \chi \quad \text{for } i = 1, \dots, n.$$

Notice that, since $Z(G) = k^*Z(G_0)$, the character group of $Z(G)$ is generated by χ. Thus, $E = \mathbb{R}\chi \oplus E_0$. In fact, χ vanishes on E_0 (in additive notation). It follows from these observations that

1. $\{\widetilde{\chi}_1, \dots, \widetilde{\chi}_{n-1}, \chi\}$ spans E;
2. if $x \in T$ lies in $T_0 \subset T$, then $\widetilde{\chi}_i|_{T_0}(x) = \chi_i(x)$;
3. $\{\widetilde{\chi}_1, \dots, \widetilde{\chi}_{n-1}\}$ spans E_0.

Let W denote the Weyl group $N_G(T)/T$ of the pair (G, T). Then the Weyl group of the maximal torus T_0 of the semisimple part $G_0 \overset{\rho_0}{\cong} (G, G)$ is isomorphic to W, also.

Lemma 3. [Proposition 3.5 in [23]] We preserve the preceding notation. Let M denote the J-irreducible monoid associated with (ρ_0, V). If λ is the T_0-highest weight of (ρ_0, V), then the set of weights of T is contained in the rational convex hull $\mathcal{P}$ of the finite set

$$\{w \cdot (\chi + \lambda) : \; w \in W\} = \{\chi + w \cdot \lambda : \; w \in W\}.$$

Furthermore, $\sigma := \mathbb{R}^+\chi_1 + \cdots + \mathbb{R}^+\chi_n \subseteq \mathbb{R}^n$ is a rational polyhedral cone over $\mathcal{P}$. In this case, σ is called the cone of M.

Example 4. Let G_0 stand for SL_n, the special linear group consisting of $n \times n$ matrices with determinant one, and let T_0 denote the maximal torus of diagonal matrices in G_0. The *first fundamental representation* $\rho_0 : G_0 \to \mathrm{GL}(k^n) \simeq \mathrm{GL}_n$ is defined by

$$\rho_0(x) = x, \quad \text{for all} \quad x \in G_0.$$

Clearly, $G = k^* \rho_0(G_0)$ is equal to $\mathrm{GL}(k^n)$, and therefore, $T = \mathbf{T}$. It is straightforward to verify that the weight ω_1[1] defined by

$$\omega_1 := \varepsilon_1 - \frac{1}{n}(\varepsilon_1 + \cdots + \varepsilon_n) \tag{2.4}$$

is the T_0-highest weight of (ρ_0, k^n). Here, ε_i's are the standard coordinate functions on T. Note that

- $\{\chi_1 - \chi, \ldots, \chi_{n-1} - \chi, \chi\}$ is a basis for $X(T)$;
- $\{\widetilde{\chi}_i = \chi_i - \chi : \; i = 1, \ldots, n-1\}$ is a basis for $X(T_0)$.

Given a positive integer $n \in \mathbb{N}$, we use the notation $[n]$ to denote the set $\{1, \ldots, n\}$. The symmetric group of permutations on $[n]$, denoted by S_n, is the Weyl group of both (GL_n, T) and (SL_n, T_0). It acts on χ_i's by permuting their indices;

$$\pi \cdot \chi_i = \chi_{\pi^{-1}(i)} \quad \text{for all} \quad \pi \in S_n, \quad \text{and} \quad i = 1, \ldots, n. \tag{2.5}$$

We re-write the n-th coordinate function in the following way:

$$\begin{aligned} \varepsilon_n &= n\chi - (\varepsilon_1 + \cdots + \varepsilon_{n-1}) \\ &= (\chi - \varepsilon_1) + \cdot + (\chi - \varepsilon_{n-1}) + \chi. \end{aligned}$$

Thus, $\widetilde{\chi}_n = \varepsilon_n - \chi = -(\widetilde{\chi}_1 + \cdots + \widetilde{\chi}_{n-1})$. Now it is easily seen that S_n stabilizes E_0, the $\mathbb{R}$-span of $\{\widetilde{\chi}_i : \; i = 1, \ldots, n-1\}$. Let σ denote the cone

$$\sigma = \mathbb{R}^+ \chi_1 \oplus \cdots \oplus \mathbb{R}^+ \chi_{n-1} \oplus \mathbb{R}^+ \chi_n \quad \text{in} \quad E \simeq \mathbb{R}^n.$$

Notice that ω_1 of (2.4) is equal to $\widetilde{\chi}_1$, and therefore, $\chi + \omega_1 = \chi_1$. Thus, by (2.5) the convex hull of vectors $w \cdot (\chi + \omega_1)$, $w \in S_n$ is equal to the convex hull of χ_i's, and σ is a cone over this polytope. We identify χ_i with the n-tuple $e_i = (0, \ldots, 1, \ldots, 0)$ having a single 1 at the i-th entry, and 0's elsewhere. The usual permutation action of S_n on e_i's agrees with the action of W on χ_i's, hence, we see that $\mathcal{P}$ is the regular $(n-2)$-simplex. See Figure 2.1 for $n = 5$, for which we use the projection from $\mathbb{R}^4$ to $\mathbb{R}^3$.

[1] More generally, for $i = 1, \ldots, n-1$ define

$$\omega_i = \varepsilon_1 + \cdots + \varepsilon_i - \frac{i}{n}(\varepsilon_1 + \cdots + \varepsilon_n).$$

It is a classical result by Cartan that ω_i is the highest weight of the i-th exterior power $\bigwedge^i k^n$ of the first fundamental representation of SL_n. By analogy, $\bigwedge^i k^n$ is called the i-th fundamental representation.

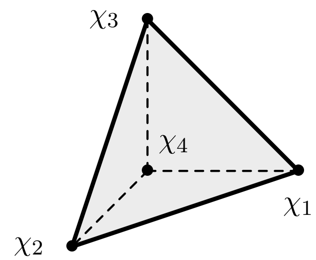

Figure 2.1: The polytope of the defining representation $\mathrm{SL}_5 \to \mathrm{GL}(k^5)$.

Example 5. Observe that $Z(\mathrm{SL}_n)$ is comprised of n-roots of 1. Let G_0 denote the adjoint group $\mathrm{SL}_n/Z(\mathrm{SL}_n)$ and (ρ_0, V) denote its adjoint representation. In other words, V is equal to $\mathfrak{sl}_n$, the space of traceless $n \times n$ matrices, and

$$\rho_0(g)(A) = gAg^{-1} \text{ for all } g \in G_0, \text{ and } A \in V.$$

Since $\ker(\rho_0) \subseteq \mathrm{SL}_n \cap \mathrm{Z}(\mathrm{Mat}_n)$, ρ_0 is faithful. As before, let G denote the extended group $k^*\rho_0(G_0) \subset \mathrm{GL}(V)$, and let T denote the maximal torus of G containing T_0. Here, $T_0 \subset G_0$ is the (image of the) maximal torus consisting of diagonal matrices. Note that $G \simeq \mathrm{GL}_n$.

Let $[n]^2$ denote the set $[n] \times [n]$, and let $([n]^2)^*$ denote the set $[n]^2 - \{(n,n)\}$. A canonical Lie algebra basis for $V = \mathfrak{sl}_n$ is given by the ordered set $B = \{F_{i,j} :\ (i,j) \in ([n]^2)^*\}$ (ordered with respect to lexicographic ordering on the indices), where

$$F_{i,j} = \begin{cases} E_{i,j} & \text{if } 1 \leq i \neq j \leq n, \\ E_{i,i} - E_{i+1,i+1} & \text{if } 1 \leq i = j \leq n-1. \end{cases}$$

Here $E_{i,j}$ is the elementary matrix with 1 at its i,j-th entry, and 0's everywhere else. We identify $\mathrm{GL}(V)$ with GL_{n^2-1} via the basis B. Let $\mathbf{T}$ denote the maximal torus of diagonal matrices in GL_{n^2-1}, and let $\varepsilon_{i,j}$ denote the coordinate function defined by

$$\varepsilon_{i,j}(\mathrm{diag}(a_1, \ldots, a_{n^2-1})) = a_{(n-1)i+j}.$$

The character group $X(\mathbf{T})$ of $\mathbf{T}$ is generated by $\varepsilon_{i,j}$'s. Let $\chi_{i,j}$ denote the restriction of $\varepsilon_{i,j}$ to T. It is a straightforward calculation to show that a generic element y from $T \subset \mathbf{T}$ is of the form $y = cx$, where

$$x = \mathrm{diag}(t_1t_1^{-1}, t_1t_2^{-1}, \ldots, t_1t_n^{-1}, \ldots, t_{n-1}t_1^{-1}, \ldots, t_{n-1}t_n^{-1}, t_nt_1^{-1}, \ldots, t_nt_{n-1}^{-1}) \tag{2.6}$$

for some diagonal matrix $\mathrm{diag}(t_1, \ldots, t_n) \in \mathrm{SL}_n$, and for some $c \in k^*$. In particular, $x \in \rho_0(T_0)$. Therefore, $\chi_{i,j}(y) = ct_it_j^{-1}$. Let χ denote the restriction to T of the (n^2-1)-th root of the product of all $\varepsilon_{i,j}$'s:

$$\chi = \left(\prod_{(i,j)\in([n]^2)^*} \chi_{i,j} \right)^{\frac{1}{n^2-1}}.$$

Thus, $\chi(y) = \chi(cx) = c = \det y$ for $y = cx \in T$, where x is as in (2.6).

For $(i,j) \in ([n]^2)^*$, we put $\widetilde{\chi}_{i,j} := \chi_{i,j} - \chi$. We know that the character group $X(T)$ is amply generated by the union $\{\widetilde{\chi}_{i,j} : 1 \le i \ne j \le n\} \cup \{\chi\}$, and that the set $\{\widetilde{\chi}_{i,j} : 1 \le i \ne j \le n\}$ generates $X(T_0)$. Both of $X(T_0)$ and $X(T)$ are sublattices of $X(\mathbf{T}) = \bigoplus_{(i,j)\in([n]^2)^*} \mathbb{Z}\varepsilon_{i,j}$.

The Weyl group $W \simeq S_n$ of (G,T) (hence of (G_0,T_0)) acts on $\widetilde{\chi}_{i,j}$'s by permuting the indices as follows:

$$w \cdot \widetilde{\chi}_{i,j} = \widetilde{\chi}_{w^{-1}i, w^{-1}j} \ \text{ for all } w \in S_n \text{ and } 1 \le i \ne j \le n.$$

Clearly, the determinant is intact by this action. Similar to the case of the previous example, we identify $\widetilde{\chi}_{i,j}$ with a suitable vector in $\mathbb{R}^n$ with entries from $\{-1,0,1\}$:

$$\widetilde{\chi}_{i,j} \leftrightsquigarrow \begin{cases} (0,\dots,0,1,0\dots,0,-1,0\dots0) & \text{if } i<j; \\ (0,\dots,0,-1,0\dots,0,1,0\dots0) & \text{if } i>j. \end{cases} \tag{2.7}$$

Here, the nonzero entries occur only at the i-th and the j-th positions. Note that there exactly $n(n-1)$ such vectors. The identification (2.7) is equivariant in the sense that the action of W on $\widetilde{\chi}_{i,j}$'s agrees with the permutation action of S_n on the entries of the corresponding vectors.

It is well known that the highest weight λ of the adjoint representation of SL_n is equal to $\omega_1 + \omega_{n-1} = \varepsilon_1 - \varepsilon_n$, where ε_i is the i-th coordinate function on T_0. Equivalently,

$$\lambda = \omega_1 + \omega_{n-1} = \widetilde{\chi}_{1,n} = \chi_{1,n} - \chi \in X(T_0) \subset X(\mathbf{T}).$$

Therefore, it follows from (2.7) that the polytope $\mathcal{P}$ (the convex hull of $W\cdot(\lambda+\chi)$) is equal to the convex hull in $\mathbb{R}^n$ of all 0/1 vectors with exactly two nonzero entries. For example, when $n = 4$ we see, after projecting from $\mathbb{R}^4$ to $\mathbb{R}^3$ that $\mathcal{P}$ is equal to the cube-octahedron depicted in Figure 2.2.

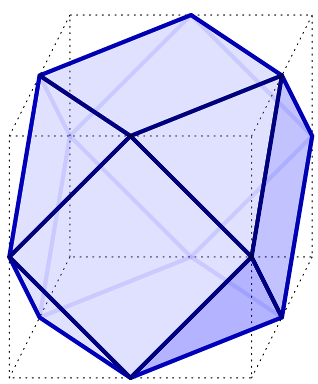

Figure 2.2: The polytope of the adjoint representation $\mathrm{SL}_4 \to \mathrm{GL}(\mathfrak{sl}_4)$.

3 Special weights; a proof of Theorem 1

Let G denote a reductive group and θ be an involution on G, Then there exists a θ-stable maximal torus T in G. Moreover, there exists a θ-stable Borel subgroup $B \subset G$ containing T. The proofs of these well-known facts and much more are recorded in [33] (see Section 7). The *fixed subtorus* T_0, and the *isotropic subtorus* T_1 associated with T are defined by

$$T_0 = \{t \in T : \ \theta(t) = t\},$$
$$T_1 = \{t \in T : \ \theta(t) = t^{-1}\}.$$

The multiplication map $T_1 \times T_0 \to T$ is an *isogeny*, that is to say a surjective homomorphism with a finite kernel.

Among all θ-stable maximal tori, we work with the one for which the dimension $l = \dim T_1$ of the isotropic subtorus is maximal. The integer l is called the rank of the symmetric space G/G^θ. Let Φ denote the set of roots (weights of the adjoint representation) of G relative to T. Passing to the Lie algebra setting by differentiation, we view Φ as a subset of the dual vector space $\mathrm{Lie}(T)^*$ of the Lie algebra of T. Since θ is an automorphism of T, it induces a linear map

$$\theta^* : \ \mathrm{Lie}(T)^* \to \mathrm{Lie}(T)^*,$$

which, in turn induces an involution on Φ. Define

$$\Phi_0 = \{\alpha \in \Phi : \ \theta^*(\alpha) = \alpha\},$$
$$\Phi_1 = \Phi - \Phi_0.$$

Lemma 6 (Lemma 1.2, [6])**.** There exists a system of positive roots $\Phi^+ \subseteq \Phi$ such that $\theta^*(\alpha) \in \Phi - \Phi^+$ for all $\alpha \in \Phi^+ \cap \Phi_1$.

We fix a set of positive roots Φ^+ as in Lemma 6. Let Δ denote the associated set of simple roots and set

$$\Delta_0 = \Phi_0 \cap \Delta,$$
$$\Delta_1 = \Phi_1 \cap \Delta.$$

Observe that $|\Delta_1| \geq \dim T_1 = l$.

It turns out there exists an ordering $\{\alpha_1, \dots, \alpha_j\}$ of the elements of Δ_1 such that the differences $\alpha_i - \theta^*(\alpha_i)$ are mutually distinct for $i = 1, \dots, l$, and for each $i \in \{l+1, \dots, j\}$, there exists an index $s \in [l]$ such that $\alpha_i - \theta^*(\alpha_i) = \alpha_s - \theta^*(\alpha_s)$. A *restricted simple root* $\overline{\alpha}$ is a weight of the form

$$\overline{\alpha} = \frac{\alpha_i - \theta^*(\alpha_i)}{2} \text{ for some } i \in [l].$$

In this case, we denote $\overline{\alpha}$ by $\overline{\alpha}_i$, and denote by $\overline{\Delta_1} = \{\overline{\alpha}_1, \dots, \overline{\alpha}_l\}$ the set of all restricted simple roots.

Suppose that $\Delta_0 = \{\beta_1, \dots, \beta_k\}$. In accordance with the partitioning $\Delta = \Delta_0 \sqcup \Delta_1$, we divide the set of fundamental weights of Δ into two disjoint subsets $\{\omega_1, \dots, \omega_j\} \sqcup \{\zeta_1, \dots, \zeta_k\}$ so that for each $i \in [j]$ the following holds:

$$(\omega_i, \beta_s^\vee) = 0 \text{ for } s = 1, \dots, k, \text{ and } (\omega_i, \alpha_r^\vee) = \delta_{i,r} \text{ for } r = 1, \dots, j.$$

Similarly for ζ_i's. As it is shown in [6] (Page 5 and 6), θ^* induces an involution $\widetilde{\theta}$ on the indices $\{1, \dots, j\}$ such that $\theta^*(\omega_i) = -\omega_{\widetilde{\theta}(i)}$. Thus, we arrive at a crucial definition for our purposes:

Definition 7. A dominant weight λ of G is called special (or, θ-special), if $\theta^*(\lambda) = -\lambda$. If (ρ, V) is an irreducible representation with a θ-special highest weight, then we call ρ a θ-special representation of G.

Let G_0 be a semisimple, adjoint type linear algebraic group, and let $\theta_0 : G_0 \to G_0$ be an involution on G_0. We choose a θ_0-stable maximal torus T_0 in G_0. Let λ be a special, dominant weight with the corresponding irreducible representation (ρ_0, V). As before, we define the reductive group G by setting $G = k^*\rho_0(G_0) \subset \mathrm{GL}(V)$. We claim that there exits an "extension" of θ_0 to G. To this end we define

$$\theta(c\rho_0(g)) = c^{-1}\rho_0(\theta_0(g)),\ g \in G_0,\ c \in k^*. \tag{3.1}$$

To prove that θ is well defined, suppose $g, g' \in G_0$ and $c, c' \in k^*$ are such that $c\rho_0(g) = c'\rho_0(g')$. Let $\alpha \in k^*$ denote cc'^{-1}. Then $\rho_0(g^{-1}g') = \alpha 1_{\mathrm{GL}(V)} \in \mathrm{GL}(V)$. Since V is irreducible, by Schur's Lemma, $g^{-1}g'$ lies in the center of G_0. In other words, $g = g'$, and therefore $c = c'$. Finally, note that

$$\theta(\theta(c\rho_0(g))) = \theta(c^{-1}\rho_0(\theta_0(g))) = c\rho_0(\theta_0(\theta_0(g))) = c\rho_0(g) \text{ for all } c \in k^* \text{ and } g \in G_0.$$

The *antiinvolution* corresponding to θ is, by definition, the composition $\theta_{an} := \theta \circ \iota$ of θ with the "inverting" morphism $\iota :\ g \mapsto g^{-1}$. The map induced by θ_{an} on the weights $X(T)$ is denoted θ^*_{an}. Then θ^* and θ^*_{an} are related to each other by

$$\theta^*_{an}(\chi) = -\theta^*(\chi) \text{ for } \chi \in X(T).$$

In particular, if $\theta^*(\lambda) = -\lambda$, then $\theta^*_{an}(\lambda) = \lambda$.

We are ready to prove Theorem 1. Let us paraphrase it for completeness: If M is a J-irreducible monoid that is obtained from a θ-special representation of G, then there exists a unique morphism $\theta_{an} : M \to M$ such that

1. $\theta_{an}(xy) = \theta_{an}(y)\theta_{an}(x)$ for all $x, y \in M$;
2. θ_{an}^2 is the identity map on M;
3. $\theta_{an}(g) = \theta(g)^{-1}$ for all $g \in G$.

Proof of Theorem 1. Since θ_{an} agrees with θ (after composing with ι, of course) on G, the uniqueness is clear. We are going to show that θ_{an} extends to whole J-irreducible monoid M.

Let T_0 denote the maximal torus of G_0 such that $T = k^*\rho(T_0)$, and let $\langle \Pi(\rho) \rangle$ denote the submonoid of $X(T)$ generated by the weights $\Pi(\rho)$ of T_0. The coordinate ring of affine torus embedding $\overline{T}$ is equal to the monoid-ring $R = k[\langle \Pi(\rho) \rangle]$. Therefore, $\overline{T} = \text{Spec}(R)$. On the other hand, from Lemma 3.1 we know that $\Pi(\rho)$ is contained in the convex hull $\mathcal{P}$ of $W \cdot (\lambda + \chi)$, where χ is the n-th root of the determinant on $\text{GL}(V)$ Since the action of θ_{an}^* on λ is trivial, hence it maps $\mathcal{P}$ onto itself, θ_{an}^* defines an involutory automorphism of the monoid $\langle \Pi(\rho) \rangle$. In particular, it induces an antiinvolution θ_{an} on $\overline{T} = \text{Spec}(R)$. Since $\theta \circ \iota = \theta_{an}$ on T, by the "extension principle" (Corollary 4.5 of [23]), there exists a unique morphism $\theta_{an} : M \to M$, which agrees with $\theta \circ \iota$ on G, and agrees with θ_{an} on $\overline{T}$.

Since $\theta_{an}^2 = id$ on G, and since G is dense on M, we see that $\theta_{an}^2 = id$ on M. Finally, since $(x, y) \mapsto \theta_{an}(xy)$ and $(x, y) \mapsto \theta_{an}(y)\theta_{an}(x)$ are morphisms from $M \times M$ into M agreeing on an open dense set $G \times G$, they agree everywhere.

□

In general, it is important to determine all special dominant weights of a given group. Obviously, if λ is a special weight, then so is 2λ. Before presenting some known examples of groups along with the lists of their special weights, we collect some information from the representation theory side of the story.

Recall that a closed subgroup $H \subseteq G$ is called spherical, if H has a dense orbit in G/B, or equivalently, B has a dense orbit in G/H. Let $k[G]$ denote the ring of regular functions on G. An irreducible representation (π, V) of G is called *H-spherical*, if the space of H-invariant vectors in V, denoted by V^H, is at most one dimensional. Assuming H is reductive, the H-spherical representations of G are precisely the irreducible representations that occur in $\mathcal{R}(G/H)$, the subring of $k[G]$ consisting of the elements $f \in k[G]$ such that $f(gh) = f(g)$ for all $h \in H$ and $g \in G$ (see [Proposition 12.2.4 [7]]).

Let G_0 be semisimple group of adjoint type and suppose θ is an involution on G_0 with the fixed point group H. Then H is a reductive subgroup of G_0. In fact, the fixed subgroup of an involution on any reductive group is reductive. Given a dominant weight λ, let $V = V_\lambda$ denote the corresponding irreducible representation of G_0 with the highest weight λ. We know from Lemma 1.5 of [6] that if V^H is at most one dimensional, then λ is a special weight. Therefore, the question of "which weights are special" has a satisfactory answer; the highest weights of all spherical representations are special.

Remark 8. Notice that the set of spherical dominant weights is a subsemigroup of all dominant weights.

By definition, a *classical group* is one of the following semisimple groups:

- $\text{SL}_n = \{g \in \text{GL}_n : \det g = 1\}$,
- $\text{SO}_n = \{g \in \text{SL}_n : gg^\top = 1\}$,
- $\text{Sp}_{2n} = \{g \in \text{SL}_{2n} : J_n(g^\top)^{-1}J_n = g\}$, where J_n is the $2n \times 2n$ block diagonal matrix $J_n = \text{diag}(J_2, \ldots, J_2)$ with $J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

There is a well known classification, due to Cartan, of involutions on semisimple groups. For the classical groups, up to inner automorphisms there are seven types of involutions in total. For the exceptional groups there are 9 involutions. See Chapter X, Section 6 of [8] for a complete list. Here we focus only on the classical groups. Following the notation of [7], in Table 1, we list the "classical" symmetric spaces along with the generators of their semigroups of spherical weights.

Classical symmetric space	Spherical fundamental dominant weights
$\mathrm{SL}_n/\mathrm{SO}_n$	$2\omega_1, 2\omega_2, \dots, 2\omega_{n-1}$
$\mathrm{SL}_{2n}/\mathrm{Sp}_{2n}$	$\omega_2, \omega_4, \dots, \omega_{2n-2}$
$\mathrm{SL}_n/S(\mathrm{GL}_p \times \mathrm{GL}_q)$	$\omega_1 + \omega_{n-1}, \omega_2 + \omega_{n-2}, \dots, \omega_p + \omega_{n-p}$ $(2l \leq n)$
$\mathrm{Sp}_{2n}/\mathrm{GL}_n$	$2\omega_1, 2\omega_2, \dots, 2\omega_n$
$\mathrm{SO}_{2n}/\mathrm{GL}_n$ $(n$ even$)$	$\omega_2, \omega_4, \dots, \omega_{n-2}, 2\omega_n$
$\mathrm{SO}_{2n}/\mathrm{GL}_n$ $(n$ odd$)$	$\omega_2, \omega_4, \dots, \omega_{n-3}, \omega_{n-1} + \omega_n$
$\mathrm{SO}_n/S(\mathrm{O}_p \times \mathrm{O}_q)$ $(n = 2k$ even$)$	$2\omega_1, 2\omega_2, \dots, 2\omega_l$ $(l < k)$ $2\omega_1, 2\omega_2, \dots, 2\omega_{l-1}, 4\omega_l$ $(l = k)$
$\mathrm{SO}_n/S(\mathrm{O}_p \times \mathrm{O}_q)$ $(n = 2k+1$ odd$)$	$2\omega_1, 2\omega_2, \dots, 2\omega_l$ $(l < k-1)$ $2\omega_1, 2\omega_2, \dots, 2\omega_{k-2}, 2\omega_{k-1} + 2\omega_k$ $(l = k-1)$ $2\omega_1, 2\omega_2, \dots, 2\omega_{k-2}, 2\omega_{k-1}, 2\omega_k$ $(l = k)$
$\mathrm{Sp}_{2n}/(\mathrm{Sp}_{2p} \times \mathrm{Sp}_{2q})$	$2\omega_1, 2\omega_2, \dots, 2\omega_{2l}$ $(2l \leq n)$

Table 1: Spherical weights for classical symmetric spaces

4 Special compactifications of G/H

To motivate our next discussion, we focus on the simplest spherical representation for $G_0/G_0^\theta = \mathrm{SL}_n/\mathrm{SO}_n$, namely, the irreducible representation of $G_0 = \mathrm{SL}_n$ corresponding to the doubled first fundamental dominant weight, $2\omega_1$. In this case, the involution on G_0 is given by $\theta(g) = (g^{-1})^\top$. Obviously, the maximal torus of diagonal matrices is θ-stable. Since the action of θ on coordinate functions is given by

$$(\theta \cdot \varepsilon_i)(x) = \varepsilon_i(\theta(x)) = \varepsilon_i(x^{-1}) = t_i^{-1} = \varepsilon_i^{-1}(x),$$

where x is the diagonal matrix $x = \mathrm{diag}(t_1, \dots, t_n)$, we see that θ^* maps the character ε_i to $-\varepsilon_i$. Here, we are using the additive notation. In particular, since the fundamental dominant weights of (G_0, T_0) are of the form $\omega_i = \sum_{s=1}^i \varepsilon_s - \frac{i}{n}\sum_{r=1}^n \varepsilon_r$, we see that any element of $X(T_0)$ is special:

$$\theta^*\left(\sum_i^{n-1} a_i\omega_i\right) = -\left(\sum_i^{n-1} a_i\omega_i\right).$$

If V_{ω_i} denotes the irreducible representation with highest weight ω_i, then the homogenous polynomials of degree k on V_{ω_i} forms an irreducible representation of G_0, and its T_0-highest weight is $k\omega_i$. In particular, $V_{\omega_1} \cong k^n$ is irreducible and the highest weight of the second symmetric power $S^2\mathbb{C}^n$ is $2\omega_1$. Concretely, this is the representation of SL_n on the space of $n \times n$ symmetric matrices, which we denote by Sym_n. The action of SL_n on Sym_n is given by

$$g \cdot A = (g^{-1})^\top A g^{-1}, \quad g \in \mathrm{SL}_n,\ A \in \mathrm{Sym}_n. \tag{4.1}$$

Note that an element $A \in \mathrm{Sym}_n$ is fixed by $H = \mathrm{SO}_n$ if and only if its is a scalar multiple of the identity transformation, and that the SL_n-orbit of the identity consists of matrices of the form $gg^\top$, $g \in \mathrm{SL}_n$.

For this paragraph only, we specialize our underlying field to $\mathbb{C}$, complex numbers. A well-known result in matrix analysis known as Takagi's factorization states that if $A \in \mathrm{Mat}_n$ is symmetric, then there exists a unitary matrix $U \in \mathrm{Mat}_n$ and a real non-negative diagonal matrix Σ such that $A = U\Sigma U^\top$. See [11]. In particular, denoting by $\Sigma^{1/2}$ the matrix obtained from Σ by taking the square root of its entries, we see that $A = (U\Sigma^{1/2})(U\Sigma^{1/2})^\top$. In conclusion, any symmetric matrix A has a decomposition of the form $A = BB^\top$ for some $B \in \mathrm{Mat}_n$. Since $\mathrm{GL}_n \simeq \mathbb{C}^*\mathrm{SL}_n$ is dense in Mat_n, we see that the set of all symmetric matrices Sym_n is the closure of the set $\{cgg^\top :\ g \in \mathrm{SL}_n, c \in \mathbb{C}^*\}$. Since $\mathcal{O} = \{gg^\top :\ g \in \mathrm{SL}_n\}$ is equal to $\mathrm{SL}_n \cdot 1_{\mathrm{Mat}_n}$ with respect to the action (4.1), we arrive at the following conclusion:

Sym_n is the embedding of the symmetric space $k^*\mathrm{SL}_n/\mathrm{SO}_n \simeq \{cgg^\top :\ g \in \mathrm{SL}_n,\ c \in k^*\}$.

In the rest of this section we generalize the above observation to all symmetric spaces. First, we recall a construction originally due to De Concini and Procesi.

Lemma 9 ([6], Lemma 1.6)**.** Let λ be a θ-special weight for (G,T), $V = V_\lambda$ be the corresponding irreducible representation, and denote by V^θ a copy of V on which G acts by $g \cdot v = \theta(g)v$. Then there is a unique (G,θ)-equivariant isomorphism

$$h : V^* \longrightarrow V^\theta \cong V. \tag{4.2}$$

Indeed, normalizing it if necessary, the isomorphism h maps v^λ, a lowest weight vector in the dual space V^* to v_λ, a highest weight vector of V.

Let h, V, v_λ and $v^\lambda \in V^*$ be as in Lemma 9. Extend the singleton $\{v_\lambda\}$ to a basis $\mathcal{V} = \{v_\lambda, v_1, \dots, v_m\}$ for V where each v_i $(i = 1, \dots, m)$ is a weight vector. Then there is the dual basis $\mathcal{V}' = \{v^\lambda, v^1, \dots, v^m\}$ with the property that if χ_i is the weight corresponding to v_i, then $-\chi_i$ is the weight corresponding to v^i, and furthermore, the weight of $h(v^i)$ is the character $-\theta^*(\chi_i)$.

Since V is self-dual by Lemma 9, there is a G-equivariant isomorphism between $\mathrm{End}(V)$ and $V \otimes V$, defined explicitly as follows: A linear map $f \in \mathrm{Hom}(V,V)$ is first mapped to

$$v^\lambda(f(v_\lambda))(v^\lambda \otimes v_\lambda) + \sum_{i,j=1}^{m} v^i(f(v_j))(v^i \otimes v_j) \in V^* \otimes V$$

via the canonical isomorphism $\mathrm{Hom}(V,V) = V^* \otimes V$, and then it is mapped to

$$v^\lambda(f(v_\lambda))(v_\lambda \otimes v_\lambda) + \sum_{i,j=1}^{m} v^i(f(v_j))(w_i \otimes v_j) \in V \otimes V, \quad \text{where } w_i = h(v^i)$$

by the isomorphism $V^* \otimes V \to V \otimes V$ sending $v' \otimes v \in V^* \otimes V$ to $h(v') \otimes v \in V \times V$.

Remark 10. By the equivariant isomorphism $\mathrm{End}(V) \to V \otimes V$, we have the assignment

$$1 = 1_{\mathrm{End}(V)} \rightsquigarrow h' = v_\lambda \otimes v_\lambda + \sum_{i=1}^{m} w_i \otimes v_i. \tag{4.3}$$

Therefore, the J-irreducible monoid $M = \overline{k^*\rho_0(G_0)} \subseteq \mathrm{End}(V)$ encloses $J = \overline{G * 1_G}$, where the $*$-action of G is defined by $g * 1 = g\theta(g)^{-1}$, and furthermore, there is an isomorphism between J and the set $\overline{G * h'}$ in $V \otimes V$.

The following facts are observed in Section 1.7 of [6]:

- The tensor $v_\lambda \otimes v_\lambda$, which is a weight vector with weight 2λ, generates a copy of $V_{2\lambda}$ in $V \otimes V$.
- $w_i \otimes v_i$'s are weight vectors with $\chi_i - \theta^*(\chi_i)$ as their corresponding weights.
- Under the canonical G-equivariant projection $p : V \otimes V \to V_{2\lambda}$ the "identity tensor" h' of (4.3) is mapped onto the vector

  $$\overline{h} = v_{2\lambda} + \sum z_i \in V_{2\lambda}, \tag{4.4}$$

  where $v_{2\lambda}$ is the highest weight vector of $V_{2\lambda}$, and z_i's are the weight vectors having distinct weights of the form $2(\lambda - \sum_{i=1}^{\ell} n_i\overline{\alpha_i})$ with $n_i \in \mathbb{Z}_{\geq 0}$.
- Since h is (G,θ)-equivariant, and p is G-equivariant, and furthermore, both of the elements $h' \in V \otimes V$ and $\overline{h} \in V_{2\lambda}$ are fixed by $H = G^\theta$.
- If, in addition to being a special weight, λ is regular, then the wonderful compactification (defined in the introduction section) of G/H is the Zariski closure $\overline{G \cdot [\overline{h}]}$ in $\mathbb{P}(V_{2\lambda})$. Here $[\overline{h}]$ is the point corresponding to the line $k \cdot \overline{h}$ in $V_{2\lambda}$.

In the light of these facts, it is natural to consider the image of the Zariski closure $\overline{P} = \overline{\{g\theta(g)^{-1} : g \in G\}}$ in $\mathbb{P}(V_{2\lambda})$. Here, the bar indicates the Zariski closure in the J-irreducible monoid M, which lies in $\mathrm{End}(V)$. Since the projection p is G-equivariant, and since a linear subspace is always closed, we see from Remark 10 that the projection p maps $\overline{P}$ isomorphically onto its image in $V_{2\lambda}$. Thus, we propose the following definition:

Definition 11. Let λ be any θ-special weight for (G_0, T_0), and let (ρ_0, V) denote the irreducible representation of G_0 with highest weight λ. As usual, let G be the group $G = k^* \cdot \rho_0(G_0) \subset \mathrm{GL}(V)$ on which θ is extended as in (3.1). Recall that $P = \{g\theta(g)^{-1} : \ g \in G\}$ is a closed embedding of G/G^θ in G. Since $P_0 = \{g\theta(g)^{-1} : \ g \in G_0\}$ is contained P, we define a *special compactification of* G_0/G_0^θ *relative to* λ as a normalization of $p(\overline{P})$ in $\mathbb{P}(V_{2\lambda})$.

Example 12. Consider ω_1 of $G_0 = \mathrm{SL}_n$ with $\theta(g) = (g^{-1})^\top$. Then $G = k^* \cdot \rho_0(G_0)$ is GL_n and $M = \mathrm{Mat}_n = \mathrm{End}(k^n)$, and the orbit $G \cdot 1_{\mathrm{Mat}_n}$ is equal to $P = k^* \cdot \{gg^\top : \ g \in \mathrm{SL}_n\}$, the set of all $n \times n$ invertible symmetric matrices. Clearly, the closure $\overline{P}$ of this set in M is equal to Sym_n. Since $V_{2\omega_1} \simeq \mathrm{Sym}_n$, the normalization in $\mathbb{P}(V_{2\omega_1})$ of the isomorphic image of $\overline{P}$ is the projective space itself;

$$\mathbb{P}(\mathrm{Sym}_n) = (\mathrm{Sym}_n - \{0\})/\sim .$$

Here, we write $[A] \sim [B]$ for $A, B \in \mathrm{Sym}_n$, if there exists $c \in k^*$ such that $A = cB$.

5 Borel orbits in Special Compactifications

In the rest of our paper we describe the parametrizing set of the orbits of a Borel subgroup in a special compactification. We start with gathering some useful facts.

On one hand, the internal structure of a Borel subgroup $B \subseteq G$ is very simple; it is the semi-direct product of a maximal torus T and a maximal unipotent subgroup U, which is normalized by T. On the other hand, the role played by B within G is huge; when G is decomposed into double cosets of B the parametrizing set is the Weyl group $W = N_G(T)/T$. Recall (from Section 2.2) that there is an extension of this decomposition to the reductive monoids. Since we need the notation from these results, we elaborate:

Lemma 13 (Generalized Bruhat-Chevalley decomposition, [24])**.** Let M be a reductive monoid with the group of invertible elements G, and let $T \subseteq B$ be a maximal torus contained in a Borel subgroup. Let $\overline{N}$ denote the closure in M of the normalizer $N = N_G(T)$ of T. If $m \in M$, then there exist $b_1, b_2 \in B$, and $\overline{n} \in \overline{N}$ such that $m = b_1 \overline{n} b_2$.

Fix an element $\overline{n} \in \overline{N}$, and let $V_{\overline{n}} \subseteq U$ denote the subgroup $V_{\overline{n}} = \{u \in U : \ u\overline{n}B \subseteq \overline{n}B\}$. Since $V_{\overline{n}}$ is a closed subgroup of U, there exists a complemantary[2] subgroup

$$U_{\overline{n},1} = \prod_{U_\alpha \nsubseteq V_{\overline{n}}} U_\alpha. \tag{5.1}$$

Similarly, let $Z_{\overline{n}} \subseteq U$ denote the closed subgroup $Z_{\overline{n}} = \{u \in U : \ \overline{n}Tu = \overline{n}T\}$, and let

$$U_{\overline{n},2} = \prod_{U_\alpha \nsubseteq Z_{\overline{n}}} U_\alpha \tag{5.2}$$

denote its complemantary subgroup. The precise structure of the orbit $B\overline{n}B$ is exhibited in the next result:

[2]Complementary in this context means that the product morphism $U_{\overline{n},1} \times V_{\overline{n}} \to U$ is an isomorphism of algebraic groups

Lemma 14 (Lemma 13.1, [26])**.** The product morphism $U_{\overline{n},1} \times \overline{n}T \times U_{\overline{n},2} \to B\overline{n}B$ is an isomorphism of varieties.

As a consequence of Lemmas 13 and 14 we have the following important observation:

Uniqueness Criterion: Given an element $m \in M$, there exist unique $u \in U_{\overline{n},1}, v \in U_{\overline{n},2}$, and $\overline{n} \in \overline{N}$ such that

$$m = u\overline{n}v. \tag{5.3}$$

We continue with the assumption that M is a reductive monoid with an antiinvolution $\theta_{an} : M \to M$. Let $H \subseteq G$ denote, as usual, the fixed subgroup G^{θ}, where $\theta : G \to G$ is the involution $\iota \circ \theta_{an}$, where ι stands for the inverse map. Recall that the $*$-action of G on M is defined in (1.2). Here, we are going to investigate the sets of $B*$-orbits in following varieties:

- the Zariski closure $\overline{P}$ in M of $P = \{g\theta(g^{-1}) : \ g \in G\} \simeq G/H$;
- the Zariski closure $\overline{Q}$ in M of $Q = \{g \in G : \ \theta(g) = g^{-1}\}$;
- and $M_Q := \{x \in M : \ \theta_{an}(x) = x\}$.

Assume from now on that T is a θ-stable maximal torus of the θ-stable Borel subgroup $B \subseteq G$. Notice in this case that the corresponding unipotent subgroup $U \subset B$ has to be θ-stable, as well. Moreover, since T is θ-stable, if n is an element from the normalizer $N = N_G(T)$, then $\theta(n)t\theta(n)^{-1} = \theta(n)\theta(t')\theta(n^{-1}) = \theta(nt'n^{-1}) \in T$. In other words, $\theta(N) = N$. It follows that the Zariski closure $\overline{N}$ is θ_{an}-stable.

Proposition 15. Any $B*$-orbit in M_Q contains an element of $\overline{N}$.

Proof. For $m \in M_Q$, as it is shown in the previous section, there exist unique $u \in U_{\overline{n},1}, v \in U_{\overline{n},2}$, and $\overline{n} \in \overline{N}$ such that $m = u\overline{n}v$. Then

$$u\overline{n}v = m = \theta_{an}(m) = \theta_{an}(v)\theta_{an}(\overline{n})\theta_{an}(u) = \theta(v)^{-1}\theta_{an}(\overline{n})\theta(u)^{-1}.$$

By the Uniqueness Criterion (5.3), we see that $\theta_{an}(\overline{n}) = \overline{n}$. Let a denote $v\theta(u)$. It is clear that $\overline{n}a$ lies in the $B*$-orbit of m. Therefore, $\overline{n}a = \theta_{an}(\overline{n}a) = \theta_{an}(a)\theta_{an}(\overline{n}) = \theta(a)^{-1}\overline{n}$ implies

$$\theta(a)\overline{n}a = \overline{n}. \tag{5.4}$$

Suppose $\overline{n} = en$ for some $e \in E(\overline{T})$ and $n \in N$. By (5.4) we see that $\theta(a)e = ena^{-1}n^{-1}$. It follows that the element $\theta(a)e = e\theta(a)e = ena^{-1}n^{-1}e = ena^{-1}n^{-1}$ is contained in the unit group of eMe. In particular, $e\theta(a)e$ lies in the unipotent radical of the Borel subgroup eBe of the $\mathscr{H}$-class of e. See Corollary 7.2 (ii) [20]. Since square roots exists in unipotent groups, we see that $(e\theta(a)e)^{1/2} = e\theta(a)^{1/2}e = (ena^{-1}n^{-1}e)^{1/2} = e(na^{-1}n^{-1})^{1/2})e = e(na^{-1/2}n^{-1})e$, which gives us the equality $\theta(a)^{1/2}e = ena^{-1/2}n^{-1}$, or equivalently $\theta(a)^{1/2}\overline{n}a^{1/2} = \overline{n}$. On the other hand, $\theta(a)^{1/2}\overline{n}a^{1/2} = \theta(a)^{1/2} * (\overline{n}a)$. Therefore, $\overline{n}$ is contained in the $B*$-orbit of m.

□

There is an obvious extension to M of the map (1.1) of Vust:

$$\tau : M \longrightarrow M$$
$$\tau(x) = x\theta_{an}(x).$$

We claim that τ and θ_{an} commute with each other. Indeed, if $m \in M$, then

$$\tau(\theta_{am}(m)) = \theta_{an}(m)\theta_{an}(\theta_{an}(m)) = \theta_{an}(m\theta_{an}(m)) = \theta_{an}(\tau(m)).$$

Therefore, if $A \subseteq M$ is θ_{an}-stable, then so are $\tau(A)$ and $\tau^{-1}(A)$. In particular, $\tau^{-1}(\overline{N})$ is θ_{an}-stable.

Lemma 16. *$T \times H$ acts on $\tau^{-1}(\overline{N})$ by $(t, h) \cdot m = tmh^{-1}$.*

Proof. Let $(t, h) \in T \times H$, and let $m \in \tau^{-1}(\overline{N})$ (hence $m\theta_{an}(m) \in \overline{N}$). It is enough to check that $\tau(tmh^{-1})T = T\tau(tmh^{-1})$. Let $t_0 \in T$. Then

$$\tau(tmh^{-1})t_0 = tm\theta_{an}(m)\theta(t^{-1})t_0. \tag{5.5}$$

Since $tm\theta_{an}(m)\theta(t^{-1}) \in \overline{N}$, the right hand side of (5.5) is equal to $t_0'tm\theta_{an}(m)\theta(t^{-1})$ for some $t_0' \in T$. □

Theorem 17. *The following sets are in bijection with each other;*

1. *$B*$-orbits in M_Q, and*

2. *$T \times H$-orbits in $\tau^{-1}(\overline{N}) \subset M$.*

Proof. We claim that $B \times H$-orbits in M are in one-to-one correspondence with $B*$-orbits in M_Q. Indeed, if $a \in M$, then

$$\tau(bah^{-1}) = bah^{-1}\theta_{an}(bah^{-1}) = ba\theta_{an}(a)\theta_{an}(b) = b * \tau(a).$$

Conversely, for $x \in M_Q$, $\tau(BaH) = B * x$ for any $a \in M$ such that $x = \tau(a)$.

By Proposition 15, we know that each $B*$-orbit intersect $\overline{N}$ non-trivially. Suppose $\overline{n}_1$ and $\overline{n}_2$ are two elements of $\overline{N}$ representing the same $B*$-orbit. Then there exists $b \in B$ such that $\overline{n}_1 = b\overline{n}_2\theta_{an}(b)$. By the Uniqueness Criterion, we must have $b = t \in T$ for some $t \in T$. Moreover, for any $t \in T$, $B * (t\overline{n}\theta_{an}(t)) = B * \overline{n}$. Therefore, $B*$-orbits in M_Q are parametrized by the $T*$-orbits in $\overline{N} \cap M_Q$. But these orbits are in one-to-one correspondence with $T \times H$ orbits in $\tau^{-1}(\overline{N})$.

□

Now we are ready to prove our second main result, which states that the following sets are finite and they are in bijection with each other:

1. $B*$-orbits in $\overline{Q}$ (respectively, $B*$-orbits in $\overline{P}$),

2. $T \times H$-orbits in $\tau^{-1}(\overline{N} \cap \overline{Q})$ (respectively, $T \times H$-orbits in $\tau^{-1}(\overline{N} \cap \overline{P})$).

Proof of Theorem 2. Finiteness follows from the fact that the sets $\overline{P}$ and $\overline{Q}$ are spherical varieties. The rest is immediate from Theorem 17 and the fact that $P \subseteq Q \subseteq M_Q$ are $B*$-stable. $\square$

We finish this section by presenting two important examples.

Example 18. Going back to Example 12, we see that $Q = \{g \in \mathrm{SL}_n : \ g = g^\top\}$ of $\mathrm{SL}_n/\mathrm{SO}_n$ coincides with $P = \{gg^\top : \ g \in \mathrm{SL}_n\}$, and therefore, $\overline{P} = \overline{Q}$ in $M = \mathrm{Mat}_n$ is equal to Sym_n. Also, the unique (anti)-extension $\theta_{an} : \mathrm{Mat}_n \to \mathrm{Mat}_n$ of $\theta(g) = (g^\top)^{-1}$, $g \in \mathrm{GL}_n$ is $\theta_{an}(x) = x^\top$, $x \in \mathrm{Mat}_n$. Therefore, M_Q is equal to $\mathrm{Sym}_n \subset \mathrm{Mat}_n$, also. Finally, we know from [34] that $B*$-orbits in Sym_n are parameterized by $n \times n$ *partial involutions*, that is to say, symmetric 0/1 matrices with at most one 1 in each row and each column.

Example 19. Let $V = \bigwedge^2 k^{2n}$ denote the second exterior power of k^{2n}. Then, according to Table 1, (ρ_0, V) is the irreducible representation of $G_0 = \mathrm{SL}_{2n}$ with special T_0-highest weight ω_2. Here, T_0 is the maximal torus consisting of diagonal matrices in G_0, and the involution is $\theta'(g) = -J(g^{-1})^\top J$, where J is the $2n \times 2n$ block diagonal matrix

$$J = \mathrm{diag}(J_2, \dots, J_2) \ \text{ with } J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Explicitly V is the space of $2n \times 2n$ skew-symmetric matrices, and the action of G_0 on V is

$$g \cdot A = (g^{-1})^\top A g^{-1}.$$

For notational ease, let us denote the operator $\rho_0(g) \in \mathrm{GL}(V)$, $g \in G_0$ by ϕ_g. Note that $\phi_J^2 = 1_{\mathrm{GL}(V)}$. It is not difficult to show that ρ_0 is faithful, and that the extension of θ' to $G = k^* \cdot \rho_0(G_0)$ is given by $\theta(c\phi_g) = c\phi_J \phi_{(g^{-1})^\top} \phi_J$, where $c \in k^*$. Next we are going to take a closer look at the (closures of the) varieties $P = \{y\theta(y)^{-1} : \ y \in G\}$ and $Q = \{y \in G : \ y^{-1} = \theta(y)\}$.

If $y = c\phi_g \in Q$ for some $g \in G_0$, and $c \in k^*$, then

$$c^{-1}\phi_g^{-1} = \theta(y) = c\phi_J \phi_{(g^{-1})^\top} \phi_J, \quad \text{or, equivalently} \quad 1_V = c^2 \phi_{gJ(g^{-1})^\top J}.$$

Since ρ_0 is faithful, $c = 1$, and $gJ(g^{-1})^\top J = 1_{G_0}$, or $g^{-1} = J(g^{-1})^\top J$. In other words, Q is isomorphic to $Q_0 = \{g \in \mathrm{SL}_{2n} : \ \theta'(g) = g^{-1}\}$. On the other hand, it is known that Q_0 is equal to $P_0 = \{g\theta'(g^{-1}) : \ g \in \mathrm{SL}_{2n}\}$. See Section 11.3.5 of [7]. Since the image of P_0 under ρ_0 is equal to P, we see that $P = Q$, and therefore, P is a closed subvariety of G. Let θ_{an} be the unique antiinvolution extension of θ to the monoid M of (ρ_0, V). Then the closure M_Q of Q in M is equal to $M_Q = \{y \in M : \ \theta_{an}(y) = y\}$.

Next, we compute the parametrizing set of $B*$-orbits in $M_Q = M_P$. To this end, we determine the normalizer of T in G. We claim that $N_G(T) = k^* \rho_0(N_{G_0}(T_0))$. Indeed, let $x = c\rho_0(g) \in G$ be an element from the normalizer of T, and let $t \in T$. Since $t = d\rho_0(t')$ for some $t' \in T_0$ and $d \in k^*$, we have $xtx^{-1} = d\rho_0(gt'g^{-1}) \in T$, or equivalently, $gt'g^{-1} \in T_0$.

Thus, t lies in $k^*\rho_0(N_{G_0}(T_0))$. The other inclusion is obvious. We continue with analyzing a typical element of $N_G(T)$. Assume that g is a monomial matrix, that is to say, every row and every column have exactly one nonzero entry. We are going to prove that, once we fix a basis, $\rho_0(g) = \phi_g$ is a monomial matrix, also. For this purpose, we choose the following basis

$$F_{i,j} = E_{i,j} - E_{j,i} \qquad 1 \leq i < j \leq 2n,$$

where $E_{i,j}$'s are the elementary matrices. Suppose that the inverse of $g \in G_0$ is the matrix $g^{-1} = (g_{k,l})_{k,l=1}^{2n}$. Obviously, g^{-1} is a monomial matrix, as well. Since $\rho_0(g) \cdot E_{i,j} = (g^{-1})^\top E_{i,j} g^{-1} = (g_{i,k}g_{j,l})_{k,l}^{2n}$, we see that

$$g \cdot F_{i,j} = \rho_0(g) \cdot F_{i,j} = (g_{i,k}g_{j,l} - g_{j,k}g_{i,l})_{k,l}^{2n}. \tag{5.6}$$

We continue with a special case of our claim by assuming that g is a diagonal matrix. Then the k,l-th entry (with $k < l$) of $g \cdot F_{i,j}$ is nonzero if and only if $i = k$ and $j = l$. In this case, $g \cdot F_{i,j} = g_{i,i}g_{j,j}F_{i,j}$. Thus, the matrix representing $\rho_0(g)$ is the $n(n-1) \times n(n-1)$ diagonal matrix $\mathrm{diag}(s_{1,2}, s_{1,3}, \dots, s_{n-1,n})$ with $s_{i,j} = g_{i,i}g_{j,j}$. Now, more generally, assume that g is a monomial matrix. Then the k,l-th entry (with $k < l$) $g_{i,k}g_{j,l} - g_{j,k}g_{i,l}$ of $g \cdot F_{i,j}$ is nonzero if and only if one of the following is true;

i) the entries of g^{-1} at its i,k-th and the j,l-th positions are nonzero at the same time, or

ii) the entries of g^{-1} at its i,l-th and the j,k-th positions are nonzero at the same time.

Observe that i) and ii) do not hold true at the same time. Observe also that, for each $i < j$ there exists a unique pair (k,l) with $k < l$ such that either i) is true, or ii) is true. Therefore, if g^{-1} is a monomial matrix, then

$$g \cdot F_{i,j} = \begin{cases} g_{i,k}g_{j,l}F_{k,l} & \text{if } g_{i,k}g_{j,l} \neq 0, \\ g_{i,l}g_{j,k}F_{k,l} & \text{if } g_{i,l}g_{j,k} \neq 0. \end{cases} \tag{5.7}$$

It follows that if g is a monomial matrix, then so is the matrix of $\rho_0(g) = \phi_g$.

Now, let $x \in M$ be an element from $\overline{N_G(T)}$. Since the elements of $\overline{N_G(T)}$ are obtained from those of $N_G(T)$ by taking limits (in the algebraic sense), we see $x \cdot F_{i,j}$ is either identically zero, or it is a scalar multiple of $F_{k,l}$ for some k,l as in (5.7). In other words, x is obtained from the image of a monomial matrix in G_0 by replacing some of its entries by zeros.

It is well know that the invertible symmetric monomial matrices modulo the maximal torus of diagonal matrices represent the $B*$-orbits in Q, and furthermore, the finite set of orbit representatives is in bijection with the fixed point free involutions of the symmetric group S_{2n} (see [28]). Thus, in our case, the representing matrices are those that are obtained from the fixed point free monomial matrices by replacing some of the nonzero entries by zeros. These are precisely the "partial fixed point free involutions," introduced in [5]. See also [3].

6 Final Remarks

Given a reductive monoid M with an antiinvolution θ_{an}, we now have the notion of a *symmetric submonoid*

$$M^{an} := \{m \in M : m\theta_{an}(m) = 1_M\}. \tag{6.1}$$

Observe that the identity element 1_M of M is the identity element 1_G of G. Therefore, $\theta_{an}(1_M) = \theta_{an}(1_G) = \theta_{an}(1_G)\theta_{an}(1_G)$, hence $\theta_{an}(1_M) = 1_M$. In other words, $1_M \in M^{an}$. Also, if $m_1, m_2 \in M^{an}$, then

$$m_1 m_2 \theta_{an}(m_1 m_2) = m_1 m_2 \theta_{an}(m_2)\theta_{an}(m_1) = m_1 \cdot 1_M \cdot \theta_{an}(m_1) = 1_M.$$

Therefore, $m_1 m_2 \in M^{\theta_{an}}$. Note that if an element $g \in G$ lies in M^{an}, then $1_G = g^{-1}\theta_{an}(g^{-1}) = g^{-1}\theta(g)$, hence $\theta(g) = g$. In other words, the group of invertible elements of M^{an} is the fixed subgroup $H = G^{\theta}$. The above argument provides us with an effective way of producing new linear algebraic monoids, namely, one for each antiinvolution θ_{an} on M.

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