The cross section of a spherical double cone

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Abstract

We investigate the poset of $SL(n)$ orbit closures in the product of two partial flag varieties. In particular we prove that it is always a lattice.

Keywords: Spherical double cones, partial flag varieties, ladder posets.
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1 Introduction

Let G be a reductive algebraic group and let X be a G variety. X is called a spherical G variety, or G-spherical, if there exists a Borel subgroup of G with a dense open orbit in X.

Let $P_1, \dots, P_k \subset G$ be a list of parabolic subgroups containing the same Borel subgroup B. The product

$$X = G/P_1 \times \cdots \times G/P_k \tag{1}$$

is a G variety via diagonal action. Determining when X is a spherical G variety carries important information for representation theory, in particular for understanding multiplicity free representations of G. In his ground breaking article [5], Littelmann initiated the classification problem and gave a list of all possible pairs of maximal parabolic subgroups P_1, P_2 such that $G/P_1 \times G/P_2$ is G-spherical. In [6], for group $G = SL(n)$ and in [7] for $G = Sp(2n)$, Magyar, Weyman, and Zelevinski classified the parabolic subgroups $P_1, \dots, P_k$ such that the product $X = G/P_1 \times \cdots \times G/P_k$ is G-spherical. It turns out, according to [6], for spherical X, the number of factors is at most 3, and $k = 3$ occurs in only special cases. Therefore, the gist of the problem lies in the case $k = 2$. This case is settled in full detail by Stembridge in [10]. More precisely, in [10] Stembridge gave a complete list of all parabolic subgroups $P_i \subset G$, $i = 1, 2$, where G is a semisimple Lie group of any type such that $G/P_1 \times G/P_2$ is spherical.

Let us call a partial flag variety of the form G/P, a G-flag variety. Of course, there is no particular reason for confining ourselves to the products of G-flag varieties only. Indeed, there are very important variations of this theme. Let $K \subset G$ be a symmetric subgroup of a connected simple algebraic group. (A subgroup $K \subset G$ is called symmetric if there exists an involutory automorphism $\theta : G \to G$ such that $K = \{g \in G : \theta(g) = g\}$.) Let B_K be a Borel subgroup of K and let $P \subset G$ be a parabolic subgroup. The natural related question is

Question: What are the conditions on (G, K, P) so that the "$G \times K$-flag variety" $G/P \times K/B_K$ is G-spherical?

It turns out that a classification of such triplets (G, K, P) is equivalent to the classification of K-spherical G-flag varieties. A proof of the equivalence, as well as classification of these triplets is given in [3]. More recently, towards the goal of better understanding vector valued orthogonal polynomials, van Pruijssen [12] has extended the classification in [3] to the case when H is an arbitrary connected reductive subgroup. See also [8]. Let us also mention that, we recently managed to classify the triplets (G, K, P) where K is a symmetric subgroup, P is a parabolic subgroup and $G/K \times G/P$ is a spherical G-variety, [1].

From representation theoretic point of view, the first mentioned classification problem amounts to understanding of the ring of invariants of a maximal unipotent subgroup in the coordinate ring of $X = G/P_1 \times G/P_2$. This problem, in turn, is closely related to the combinatorics of the G orbits in X (see [5]). Our goal in this note is to prove the following result on G-orbits.

Theorem 1.1. Let G denote the special linear group $SL(n)$ and let P_1 and P_2 be two parabolic subgroups. If the G action on $G/P_1 \times G/P_2$ is spherical, then the inclusion poset on G orbit closures is a lattice.

More precise description of these lattices is given in Section 4.

2 Preliminaries

For simplicity we assume that G is simple and simply connected. Let $B = UT \subset G$ be a Borel subgroup with the maximal unipotent subgroup $U \subset B$ and a maximal torus $T \subset B$.

2.1

Let Φ be the root system determined by (G, T) let $\Delta = \{\alpha_1, \dots, \alpha_n\} \subset \Phi$ denote its system of simple roots relative to B and let $\omega_1, \dots, \omega_n$ denote the associated fundamental dominant weights, which form a basis for the weight lattice. In particular, for each $j = 1, \dots, n$, there exists an irreducible representation of G, denoted by $V(\omega_j)$, and a line $\ell \subset V(\omega_j)$ such that the stabilizer of $\ell \in \mathbb{P}(V(\omega_j))$ is a maximal parabolic subgroup $P_j \subset G$. In particular we have the canonical embedding $G/P_j \hookrightarrow \mathbb{P}(V(\omega_j))$.

Let $\mathscr{C}_j \subset V(\omega_j)$ denote the cone over G/P_j and set $\mathscr{C}_{i,j} := \mathscr{C}_i \times \mathscr{C}_j$. In [5], Littelmann shows that for diagonal action of U the U-invariant functions on $\mathbb{C}[\mathscr{C}_{i,j}]$ is a polynomial ring and he classifies all pairs (ω_i, ω_j) such that $\mathbb{C}[\mathscr{C}_{i,j}]^U$ is freely generated by elements of linearly independent weights. This amounts to classifying spherical products $G/P_i \times G/P_j$ of partial flag varieties attached to fundamental weights.

2.2

Suppose G acts on two irreducible varieties X_1 and X_2, and $x_i \in X_i$, $i = 1, 2$ be points in general position. If $G_i \subset G$ denotes the stabilizer subgroup of x_i in G, then $\mathrm{Stab}_G(x_1 \times x_2)$ coincides with the stabilizer in G_1 of a point in general position from G/G_2 (or, equivalently, with the stabilizer in G_2 of a point in general position from G/G_1). Consequently, studying G orbits in a spherical product $G/P_1 \times G/P_2$ reduces to the study of P_1 orbits in G/P_2.

Remark 2.1. 1. The parabolic group P_1 acts on the product $G \times G/P_2$ by $g \cdot (g_1, xP_2) = (g_1 g^{-1}, gxP_2)$. This action is free; we denote the quotient by $G \times^{P_1} G/P_2$. Thus, two points $x, y \in G \times G/P_2$ projects to represent the same point $[x] = [y]$ in $G \times^{P_1} G/P_2$ if they are of the form $x = (g_1, x_1 P_2)$, $y = (g_2, x_2 P_2)$ and there exists $g \in P_1$ such that $g_1 = g_2 g^{-1}$ and $x_1 P_2 = g x_2 P_2$.

2. Extending the action of P_1 on $G \times G/P_2$, the whole group G acts on $G \times^{P_1} G/P_2$. Then the map

$$G \times^{P_1} G/P_2 \to X = P_1 \backslash G \times G/P_2, \qquad [(g, qP_2)] \mapsto (P_1 g^{-1}, gqP_2) \tag{2}$$

is a G equivariant isomorphism.

So our objective is to understand P_1 orbits in G/P_2. It is well known that these orbits are parametrized by(W_1, W_2)-double cosets in W, where W_1, W_2, and W are the Weyl groups of P_1, P_2 and G, respectively. The Bruhat order on $W_1 \backslash W / W_2$ is well-known. In fact, it has various characterizations (see [11, Proposition 1.8]). Let

$$\pi : W \to W_1 \backslash W / W_2$$

denote the canonical projection onto double cosets. It turns out that the preimage in W of every double coset in $W_1 \backslash W / W_2$ is an interval with respect to Bruhat order, hence it has a unique maximal and a unique minimal element. Moreover, if $[w_1], [w_2] \in W_1 \backslash W / W_2$ are two double cosets represented by the maximal elements w_1, w_2 of the corresponding intervals, then $[w_1] \leq [w_2]$ in $W_1 \backslash W / W_2$ if and only if $w_1 \leq w_2$ in W. (See [4].) Although it looks as if this observation settles the question of inclusion order on closures of G orbits, understanding the exact nature of this poset is very much desirable for better understanding the products.

2.3

Without loss of generality we assume that T is contained in both of the parabolic subgroups P_1 and P_2. Also, let B be a Borel subgroup such that $T \subset B \subset P_1 \cap P_2$. Hence, P_2

and P_1 are uniquely determined by subsets J and I, respectively, of the set of simple roots $\Delta = \Delta(G, B, T)$ of Φ. In particular, the Weyl (sub)groups corresponding to P_2 and P_1, denoted by W_J and W_I, respectively, are generated by the simple reflections in the sets I, J, respectively. We reserve the letter R for the Coxeter generators $R = R(\Delta)$ of W. (Thus, I and J are subsets of R, rather than subsets of Δ; however, whenever it is convenient we identify I (and J) with the corresponding set $\widetilde{I} \subset \Delta$ of simple roots. We prefer to use this notation since the action of W on these sets is more clear in the context of Weyl groups.) Also, from now on we are going to write P_I and P_J instead of P_1 and P_2, respectively.

The set of distinguished (W_I, W_J)-double coset representatives is defined as follows. Let $[w]$ be a representative of a double coset in $W_I \backslash W / W_J$ such that $\ell(w)$ is as small as possible. It turns out that such $w \in W$ are characterized by $w \in {}^I W \cap W^J$, where ${}^I W$ is the set of minimal length coset representatives for $W_I \backslash W$. We denote ${}^I W \cap W^J$ by $X_{I,J}^-$. Set $H = I \cap wJw^{-1}$. Then $uw \in W^J$ for $u \in W_I$ if and only if u is a minimal length coset representative for W_I/W_H. In particular, every element of $W_I w W_J$ has a unique expression of the form uwv with $u \in W_I$ is a minimal length coset representative of W_I/W_H, $v \in W_J$ and $\ell(uwv) = \ell(u) + \ell(w) + \ell(v)$.

Another characterization of the sets $X_{I,J}^-$ is as follows. For $w \in W$, the *right ascent set* is defined as

$$\mathrm{Asc}_R(w) = \{s \in R : \; \ell(ws) > \ell(w)\}.$$

The *right descent set*, $\mathrm{Des}_R(w)$ is the complement $R - \mathrm{Asc}_R(w)$. Similarly, the *left ascent set* of w is

$$\mathrm{Asc}_L(w) = \{s \in R : \; \ell(sw) > \ell(w)\} \;\; (= \mathrm{Asc}_R(w^{-1})).$$

Then

$$\begin{aligned} X_{I,J}^- &= \{w \in W : \; I \subseteq \mathrm{Asc}_L(w) \;\text{ and } J \subseteq \mathrm{Asc}_R(w)\} && (3) \\ &= \{w \in W : \; I^c \supseteq \mathrm{Des}_R(w^{-1}) \;\text{ and } J^c \supseteq \mathrm{Des}_R(w)\} && (4) \end{aligned}$$

For our purposes we need the distinguished set of maximal length representatives for each double coset.

$$\begin{aligned} X_{I,J}^+ &= \{w \in W : \; I \subseteq \mathrm{Des}_R(w^{-1}) \;\text{ and } J \subseteq \mathrm{Des}_R(w)\} && (5) \\ &= \{w \in W : \; I^c \supseteq \mathrm{Asc}_R(w^{-1}) \;\text{ and } J^c \supseteq \mathrm{Asc}_R(w)\} && (6) \end{aligned}$$

For a proof of this characterization of $X_{I,J}^+$ see [2, Theorem 1.2(i)].

Remark 2.2. The Bruhat orders on $X_{I,J}^-$ and $X_{I,J}^+$ are isomorphic. Indeed, the spherical products $P_I \backslash G \times G / P_J$ and $P_I^{op} \backslash G \times G / P_J^{op}$ are isomorphic, where P_I^{op} stands for the opposite parabolic to P_I.

3 Tight Bruhat posets

We already know the classification of parabolic subgroups P_I, P_J corresponding to fundamental dominant weights such that the diagonal G action on $G/P_I \times G/P_J$ is spherical. Said differently, we know all pairs (I, J) of subsets of Coxeter generators $R \subset W$ such that

- $|I| = |J| = |R| - 1$, and
- $G/P_I \times G/P_J$ is G spherical.

In particular, under the maximality assumption on subsets I and J, the poset of G-orbit closures is always a chain, see [5, Proposition 3.2]. In the light of our remarks above, this is equivalent to showing that $W_I \backslash W / W_J$ is a chain.

As we mentioned earlier, the classification of Littelmann is extended by Stembridge to cover all pairs of subsets (I, J) in R such that $G/P_I \times G/P_J$ is G spherical. See Corollaries 1.3.A – 1.3.D, 1.3.E6, 1.3.E7, and 1.3.{E8,F4,G2} in [10].

Remark 3.1. 1. In the cases of A–D, E6, and E7, if $G/P_I \times G/P_J \neq G/B \times G/B$ is a spherical product, then at least one of I and J is maximal, that is to say, of cardinality $|R| - 1$. Without loss of generality we always choose I to be the maximal one.

Next, we review the useful concept of *tight Bruhat order*. Let $\mathbf{E}$ be a real vector space of smallest dimension containing our root system Φ. Let $\Delta \subset \Phi$ be a basis for $\mathbf{E}$ (hence, Δ is a set of simple roots), (W, R) be the associated Coxeter system. We are interested in the case when W is a Weyl group. One way to define the Bruhat-Chevalley order on W is to use the reflection representation of W as the group of isometries of $\mathbf{E}$: let $\langle \ , \rangle$ denote the W-invariant inner product on $\mathbf{E}$, and let $\theta \in \mathbf{E}$ be a vector such that $\langle \theta, \beta \rangle \geq 0$ for all $\beta \in \Phi^+$. Such a vector is called dominant. It is well-known that the stabilizer of a dominant vector is a parabolic subgroup $W_J \subset W$, where $J = \{s_\alpha \in R : \ \langle \theta, \alpha \rangle = 0\}$. Thus, as a set, the minimal length coset representatives $W^J \subset W$ of the quotient W/W_J can be identified with the orbit $W \cdot \theta$. Following Stembridge, we are going to call the orbit map $w \mapsto w \cdot \theta$ the evaluation.

A proof of the following result can be found in [9]:

Proposition 3.2. Let $\theta \in \mathbf{E}$ be a dominant vector with stabilizer W_J. The evaluation map induces a poset isomorphism between the Bruhat-Chevalley order on W^J and the orbit $W\theta$ with partial order defined by the transitive closure of the relations

$$\mu <_B s_\beta(\mu) \ \text{ for all } \beta \in \Phi^+ \text{ such that } \langle \mu, \beta \rangle > 0.$$

Now, let $I \subset R$ be a subset of the Coxeter generators for W, and let $\Phi_I \subset \Phi$ denote the root subsystem corresponding to parabolic subgroup W_I. Accordingly, let Φ_I^+ denote $\Phi^+ \cap \Phi_I$. If θ is a dominant vector and its stabilizer subgroup is W_J with $J \subset R$, then define

$$(W\theta)_I := \{\mu \in W\theta : \ \langle \mu, \beta \rangle \geq 0 \ \text{ for all } \beta \in \Phi_I^+\}. \tag{7}$$

A proof of the following result can be found in [11, Proposition 1.5].

Proposition 3.3. Let $I, J \subset R$ be two sets of Coxeter generators for W and let $\theta \in \mathbf{E}$ be a dominant vector with stabilizer W_J. Then the evaluation map induces a poset isomorphism between the (restriction of) Bruhat-Chevalley order on $X^-_{I,J}$ and $(W\theta)_I$ with partial order defined by the transitive closure of the relations

$$\mu <_B s_\beta(\mu) \text{ for all } \beta \in \Phi^+ \text{ such that } s_\beta(\mu) \in (W\theta)_I \text{ and } \langle \mu, \beta \rangle > 0.$$

Now we come to the definition of a critical notion for our proof. There is a natural partial ordering on the roots defined by

$$\nu \preceq \mu \iff \mu - \nu \in \mathbb{R}^+\Phi^+. \tag{8}$$

It turns out, when the interpretation of Bruhat-Chevalley ordering as given in Proposition 3.2 is used, there is a natural order reversing implication:

$$\mu \leq_B \nu \implies \nu \preceq \mu. \tag{9}$$

If the converse implication also holds, then the poset $W\theta$ is called tight. More precisely, a subposet $(M, \leq_B)$ of the Bruhat-Chevalley order on $(W\theta, \leq_B)$ is called tight if

$$\mu \leq_B \nu \iff \nu \preceq \mu$$

for all ν, μ in $M \subset \mathbf{E}$.

In the light of our Remark 3.1, we assume that $I \subset R$ is a maximal subset, that it is of the form $I = R - \{s\}$ for some $s \in R$. Also, we assume that there exists a dominant $\theta \in \mathbf{E}$ such that W_J is its stabilizer subgroup.

Now, by [11, Theorem 2.3], we see that if W^J is tight, then $X^-_{I,J} = X^-_{R-\{s\},J}$ is a chain. The list of tight quotients is also given in [11]; $(W^J, \leq_B)$ is tight if and only if W is of at most rank 2, or $J = R$, or one of the following holds:

- $W \cong A_n$ and $J^c = \{s_j\}$ $(1 \leq j \leq n)$ or $J^c = \{s_j, s_{j+1}\}$ $(1 \leq j \leq n-1)$,
- $W \cong B_n$ and $J^c = \{s_1\}, \{s_2\}, \{s_n\}$, or $J^c = \{s_1, s_2\}$,
- $W \cong D_n$ and $J^c = \{s_1\}, \{s_2\}$ or $J^c = \{s_n\}$,
- $W \cong E_6$ and $J^c = \{s_1\}$ or $J^c = \{s_6\}$,
- $W \cong E_7$ and $J^c = \{s_7\}$,
- $W \cong F_4$ and $J^c = \{s_1\}$ or $J^c = \{s_4\}$, or
- $W \cong H_3$ and $J^c = \{s_1\}$ or $J^c = \{s_3\}$.

Therefore, in these cases (when I is maximal and J is as in this list), then we know that $X^-_{I,J} = X^-_{R-\{s\},J}$ is a chain. Here is the list of the remaining cases under the assumption that I is of the form $R - \{s\}$ for some $s \in R$:

- $W \cong A_n$
  1. $I^c = \{s_2\}$ or $\{s_{n-1}\}$ and $J^c = \{s_p, s_q\}$ $(1 < p < p+1 < q < n)$,
  2. $|I^c| = 1$ and $J^c = \{s_1, s_j\}$ or $\{s_j, s_n\}$ with $2 < j < n-1$.
- $W \cong B_n$
  1. $I^c = \{s_n\}$ and $|J^c| = 1$.
- $W \cong C_n$
  1. $I^c = \{s_n\}$ and $|J^c| = 1$.
- $W \cong D_n$ $(n \geq 4)$
  1. $I^c = \{s_n\}$ and $J^c = \{s_i\}$ $(1 < i < n)$ or $J^c = \{s_l, s_i\}$ $(1 \leq i \leq n,\ l = 1, 2)$,
  2. $I^c = \{s_1\}$ or $\{s_2\}$ and $J^c \subsetneq \{s_1, s_2, s_n\}$ or $J^c \subset \{s_{n-1}, s_n\}$ or $J^c = \{s_{n-2}\}$. (Exclude the case $J^c = \{n\}$.)
  3. ($n = 4$ case only) $I^c = \{s_1\}$ and $J^c = \{s_2, s_3\}$ or $I^c = \{s_2\}$ and $J^c = \{s_1, s_3\}$.
- $W \cong E_6$
  1. $I^c = \{s_1\}$ or $\{s_6\}$ and $J^c = \{s_i\}$ $(i = 2, 3, 5)$ or $J^c = \{s_1, s_6\}$.

We are going to analyze the type A case in the next section.

4 Main result

Notation change: For simplicity of our notation from now on we identity a simple reflection s_i with its index i.

4.1 Case 1.

We start with the case $I^c = \{2\}$.

Let $w = w_1 \dots w_{n+1}$ be an element, in one-line notation, from $X^+_{I,J}$. Recall that

$$X^+_{I,J} = \{w \in W :\ I^c \supseteq \mathrm{Asc}_R(w^{-1}) \text{ and } J^c \supseteq \mathrm{Asc}_R(w)\}$$

The meaning of $I^c = \{2\} \supseteq \mathrm{Asc}_R(w^{-1})$ is that either $\mathrm{Asc}_R(w^{-1}) = \emptyset$, in which case w is equal to w_0, the longest permutation, or, $\mathrm{Asc}_R(w^{-1}) = \{2\}$ hence 2 comes before 3 in w. Similarly, $\mathrm{Asc}_R(w)$ cannot be empty unless $X^+_{I,J} = \{w_0\}$.

We continue with the assumption that $w \neq w_0$. Suppose $J^c = \{p, q\}$ for $1 < p < p+1 < q < n$. We are going to write L_1 for the segment $w_1 w_2 \dots w_{p-1}$, L_2 for the segment $w_p w_{p+1} \dots w_{q-1}$, and L_3 for the segment $w_q \dots w_{n+1}$. By our assumptions, all three of these segments are decreasing sequences. In particular, since 2 comes before 3 in w, 2 cannot appear in L_3. In fact, 2 and 3 cannot appear in the same segment. For convenience we are going to use bars between these segments.

First, we assume that $p = 2$. Since any element of of $X^+_{I,J}$ has descents (at least) at the positions $J = \{1, \widehat{2}, 3, 4, \ldots, \widehat{q}, \ldots, n+1\}$, the bottom element τ_0 has to be of the form

$$\tau_0 = 2\ 1\ |n+1\ n \ldots n-q+3\ \ n-q+2\ \ n-q+1 \ldots 3, \tag{10}$$

or of the form

$$\tau_0 = n+1\ n \ldots n-q+4\ \ 2\ \ 1 \mid n-q+3\ \ n-q+2 \ldots 3. \tag{11}$$

Bars between numbers indicate the possible positions of ascents. Note that the number of inversions of the former permutation is $1 + \binom{n-1}{2}$, and the rank of the latter is

$$\begin{aligned} f_n(q) &:= \left(\sum_{i=1}^{q-2} n+1-i\right) + 1 + \left(\sum_{i=q+1}^{n} n+1-i\right) \\ &= \binom{n+1}{2} + 1 - (n+1-q) - (n+1-(q-1)). \end{aligned}$$

which is always greater than the former. Therefore, the minimal element τ_0 of $X^+_{I,J}$ starts with 2 1 (as in 10).

This element has a single ascent at the 2nd position. We are going to analyze the covers of τ_0. Since an upward covering in Bruhat order is obtained by moving a larger number to the front, $n+1$ of L_2 moves into L_1 and accordingly either 2 or 1 from L_1 moves into L_2.

Now, recall that each double coset $W_I z W_J$ is an interval of W in Bruhat order and $X^+_{I,J}$ consists of maximal elements of these intervals (see [2, Theorem 1.2(ii)]). It follows from this critical observation that, to obtain a covering of τ_0, 1 has to move, and it becomes the last entry of L_2. In other words, permutation

$$\tau_1 = n+1\ 2 \mid n \ldots n-q+3\ \ 1 \mid n-q+2\ \ n-q+1 \ldots 3$$

is the unique element in $X^+_{I,J}$ that covers τ_0.

Next, we analyze the covers of τ_1 which has only two possible coverings obtained as follows: 1) 2 moves into L_2 and n moves into L_1, 2) 1 moves into L_3 and $n-q+2$ moves into L_2. The resulting elements are

$$\begin{aligned} \tau_2 &= n+1\ n \mid n-1 \ldots n-q+3\ \ 2\ \ 1 \mid n-q+2\ \ n-q+1 \ldots 3 \\ \tau_3 &= n+1\ 2 \mid n \ldots n-q+3\ \ n-q+2 \mid n-q+1 \ldots 3\ 1 \end{aligned}$$

It is not difficult to see that each of these two elements are covered by the same element, namely

$$\tau_4 = n+1\ n \mid n-1 \ldots n-q+3\ \ n-q+2\ \ 2\ \mid\ n-q+1 \ldots 3\ 1.$$

Observe that, in τ_4 the only element that can be moved is 2 and this is possible only if $q \leq n-1$. This agrees with our assumption on q. Therefore, τ_4 is covered by w_0 only (in

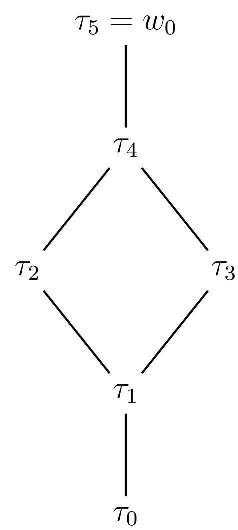

Figure 1: Bruhat order on $X^+_{I,J}$ for type A_n (i)

$X^+_{I,J}$). Note that all that is said above is independent of n as long as $p = 2$ and $3 < q < n$. Hence, our poset is as in Figure 1.

Finally for type A_n (i) with $I^c = \{2\}$, we look at the case for $p > 2$. The only difference between this and $p = 2$ case is that the first $p - 2$ terms of the elements of $X^+_{I,J}$ all start with $n+1\ \ n\ \ n-2 \dots n-p$. By induction, we reduce this case to $p = 2$ and the poset $X^+_{I,J}$ is in fact isomorphic to the one in Figure 1.

Now we assume that $I^c = \{n-1\}$ and $J = \{p, q\}$ with $2 \leq p < p+1 < q \leq n-1$. As in the previous case, for an element $w \in X^+_{I,J}$ these conditions imply that w is of the form $w = L_1|L_2|L_3$, where L_i, $i = 1, 2, 3$ are decreasing sequences of lengths $p, q-p$ and $n+1-q$, respectively, and the number $n-1$ appears before n in w. It follows that the smallest element of $X^+_{I,J}$ is of the form

$$\tau_0 = w_1 \dots w_p | w_{p+1} \dots w_q | w_{q+1} \dots w_{n+1} = n{-}1\ n{-}2 \dots n{-}q \mid n{+}1\ \ n\ \ n{-}q{-}1\ \ n{-}q{-}2 \dots 1$$

Then arguing exactly as in the previous case one sees that the poset under consideration is also of the form Figure 1.

4.2 Case 2.

We start with the case $|I^c| = 1$, $J^c = \{1, j\}$ with $2 < j < n-1$. Here, we have more variety than the previous case. Let us first analyze the sub-case $I^c = \{1\}$. Once again we are going to use bars to indicate the positions of the ascents (defined by J^c). As before, the strings between these bars are denoted by L_1, L_2, and L_3.

Suppose $w = w_1 \dots w_{n+1}$ is the smallest element τ_0 of $X^+_{I,J}$. In $w \in X^+_{I,J}$, the number 1 comes before 2. If 1 is in L_2, then we can move it to L_1 (and move the number in L_1 to L_2 at an appropriate position) without altering the ascent positions, hence we stay in $X^+_{I,J}$. At the same time, such a move reduces the length in Bruhat order, contradicting the minimality of w. Therefore, 1 has to be in L_1 to begin with. Since there are only two possible ascent positions one of which is at the 1st position and there is a 1 in it, by computing the length of the permutation w' that has a second ascent at the jth position we see that it cannot be smaller than

$$w = 1 \mid \; n+1 \;\; n \dots n-(j-1) \;\; n-j \dots 2, \tag{12}$$

in Bruhat order. Therefore, τ_0 is the element w in 12. In fact, there is a unique element in $X^+_{I,J}$ that covers τ_0 whose unique ascent is at the jth position:

$$\tau_1 = n+1 \;\; n \dots n+2-j \;\; 1 \mid n-(j-1) \;\; n-j \dots 2.$$

Now, because no other ascent is allowed, there is only one way to move up in the Bruhat order by moving a smaller entry from the string L_3 to L_2 which in fact gives the longest permutation. Therefore, the poset of $X^+_{I,J}$ is a chain of length 3:

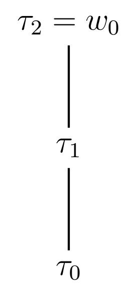

Figure 2: Bruhat order on $X^+_{I,J}$ for type A_n, $I^c = \{1\}$, $J^c = \{1, j\}$.

Now we look at the general case $I^c = \{i\}$ $(1 < i \leq n)$, $J^c = \{1, j\}$ with $2 < j < n-1$. As the number $i \in I^c$ grows (up to $\lfloor \frac{n+1}{2} \rfloor$) we get more freedom to position i and $i+1$. This makes our posets grow taller. Rather than listing each case separately, which depend on the choice of i and j, we present the general combinatorial rule that governs the structure of $X^+_{I,J}$. The underlying idea of our description is already present in the previous cases.

Our first task is to decide for the smallest element τ_0 of $X^+_{I,J}$. Once again, a generic element $w = w_1 \dots w_{n+1} \in X^+_{I,J}$ is viewed as a concatenation of three segments, $w = L_1 L_2 L_3$ where $L_1 = w_1$, $L_2 = w_2 \dots w_j$, and $L_3 = w_{j+1} \dots w_{n+1}$. Possible ascents are at the 1st and at the jth positions. At the same time, the number i appears before $i+1$ in w, therefore, i and $i+1$ are always contained in distinct segments (except in w_0). In particular, i appears either in L_1 or in L_2.

Assume that $i \in L_2$ (hence $i+1 \in L_3$) and let k denote the number in L_1. In particular, we know that $k \notin \{i, i+1\}$. By interchanging k by $k-1$ we move down in the Bruhat order and still stay in $X^+_{I,J}$, which contradicts the minimality of τ_0. Therefore, the only possibility for k is $k = 1$. But this puts us back in the case $I^c = \{1\}$, $J^c = \{1, j\}$, and there is not much ado. Therefore, we assume that $i \in L_1$. Now there are two cases each which is easy to verify:

1) $j \le i$ and τ_0 is of the form

$$\tau_0 = i \;\; i-1 \dots i-j+1 \mid n+1 \;\; n \dots i+1 \;\; i-j \;\; i-j-1 \dots 1. \tag{13}$$

2) $j > i$ and τ_0 is of the form

$$\tau_0 = i \;\; n+1 \;\; n \dots n+2-(j-i) \;\; i-1 \;\; i-2 \dots 1 \mid n+1-(j-i) \;\; n-(j-i) \dots i+1 \tag{14}$$

Now we make some obvious observations regarding how the posets climbs up in Bruhat order on $X^+_{I,J}$ starting with τ_0's as in 1) and 2). First of all, if τ_0 is as in 1), then to get a covering relation there is only one possible interchange, namely, moving $i-j+1 \in L_2$ into L_3. In this case, to maintain the descents, the number that is replaced by $i-j+1$ has to be $n+1$, which goes into the first entry of L_2. In other words, the unique $w \in X^+_{I,J}$ that covers τ_0 is

$$w = i \;\; n+1 \;\; i-1 \dots i-j+2 \mid n \dots i+1 \;\; i-j+1 \;\; i-j \;\; i-j-1 \dots 1. \tag{15}$$

It is easy to verify that there are exactly two elements that covers w:

$$w_{(2)} = n+1 \;\; i \;\; i-1 \dots i-j+2 \mid n \dots i+1 \;\; i-j+1 \;\; i-j \;\; i-j-1 \dots 1. \tag{16}$$

$$w^{(2)} = i \;\; n+1 \;\; n \;\; i-1 \dots i-j+3 \mid n-1 \dots i+1 \;\; i-j+2 \;\; i-j+1 \dots 1. \tag{17}$$

Remark 4.1. Suppose $w \in S_n$ is a permutation on $[n] := \{1, \dots, n\}$ whose one-line notation ends with the decreasing string $k \;\; k-1 \dots 2 \;\; 1$. Then any element in the upper interval $[w, w_0] \subset S_n$ has the same ending. In other words, if $w' \in [w, w_0]$, then the last k entries of w' are exactly $k, k-1, \dots, 1$ in this order. Similarly, if w begins with the decreasing string $n \;\; n-1 \dots \;\; k$ for some $k \in [n]$, then any element in the upper interval $[w, w_0] \subset S_n$ has the same beginning. So, essentially these elements form an upper interval in S_{n-k}.

By Remark 4.1 we see that the elements that are above $w_{(2)}$ in $X^+_{I,J}$ all start with $n+1$. Also, since there is no ascent at the 1st position for such elements, we see that the resulting upper interval $[w_{(2)}, w_0]$ in $X^+_{I,J}$ is isomorphic to a similar double coset poset in S_n with $I^c = \{i\}$ and $J^c = \{j\}$, hence it is a chain.

There are two covers of $w^{(2)}$; one of them, $w_{(3)}$, is an element of the interval $[w', w_0]$ (hence $w_{(3)}$ covers w' as well). The other cover of $w^{(2)}$ is

$$w^{(3)} = i \;\; n+1 \;\; n \;\; n-1 \;\; i-1 \dots i-j+4 \mid n-1 \dots i+1 \;\; i-j+3 \;\; i-j+2 \dots 1. \tag{18}$$

Now the pattern is clear; $w^{(3)}$ has exactly two covers one of which lies in $[w_{(3)}, w_0]$ and the other $w^{(4)}$ has a similar structure to $w^{(3)}$. Therefore, the resulting poset is a ladder, as depicted in Figure 3, and the chains $w^{(p)}$ and $w_{(p)}$, $p \geq 3$ will climb up to meet first time either at w_0 or at

$$w^{(m+1)} = w_{(m+1)} = n+1 \;\; n \ldots n+1-(j-3) \;\; i \mid n+1-(j-4) \ldots \widehat{i} \ldots 2 \;\; 1 \tag{19}$$

In the latter case, of course, w_0 is the unique cover of $w^{(m+1)} = w_{(m+1)}$. In particular, the hight of our poset does not exceed j.

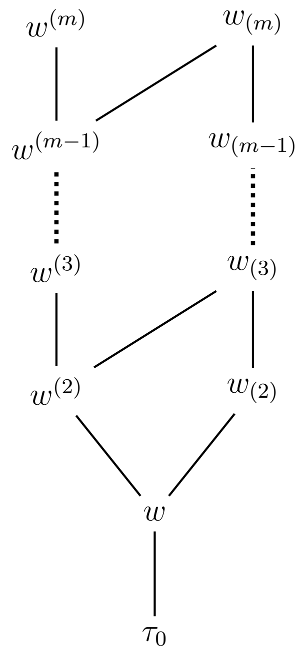

Figure 3: Bruhat order on $X^+_{I,J}$ for type A_n, $I^c = \{2\}$, $J^c = \{1, j\}$

The remaining case is $I^c = \{i\}$ and $J^c = \{j, n\}$ with $2 < j < n-1$. The proofs and the results are identical to the previous case. The poset is either a chain when $I^c = \{1\}$ and j arbitrary, or it is isomorphic to one of the ladder posets (depicted below).

In summary, we proved our main result.

Theorem 4.2. The poset of G orbit closures in the spherical product $G/P_I \times G/P_J$ is either a chain or one of the "ladder lattices" as depicted in Figure 4.

We anticipate that the following is true for other types: if $|J^c| = 1$, then the poset of G orbit closures is a chain, otherwise it is one of the ladder lattices as in Figure 4. These cases will be handled in an upcoming manuscript.

5 Final remarks

The P_I orbits in G/P_J are parametrized by W_I orbits in W/W_J. Now that we understand how these orbits fit together in Bruhat order, next we are going to describe the Bruhat order

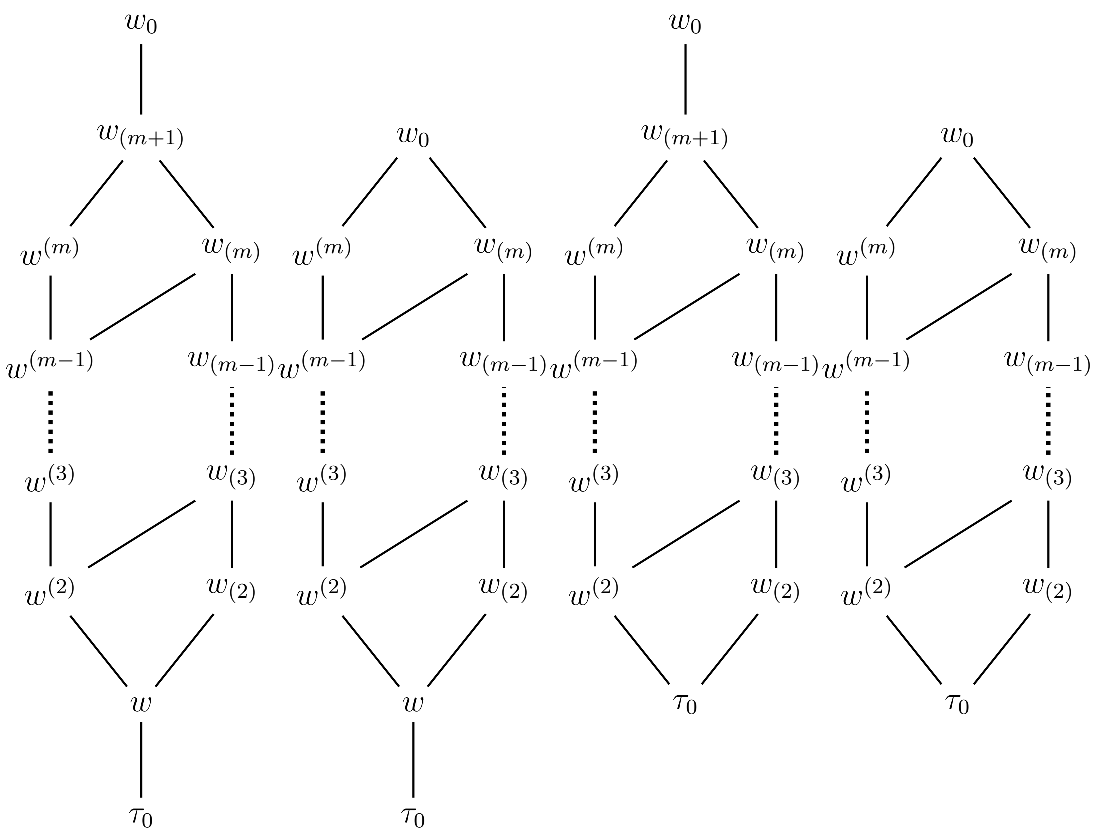

Figure 4: Possible ladder posets describing Bruhat orders on $X^+_{I,J}$ for type A_n

on B orbits in $X = P_I\backslash G \times G/P_J$. By using the G equivariant isomorphism defined in 2 and the Bruhat decomposition $P_I = BW_IB$, we see that B orbits in X are parametrized by the union of the elements of W_I orbits in W/W_J.

We already know that the set $X^-_{I,J}$ is a parametrization of (W_I, W_J) double coset representatives in W. In particular, if $w \in X^-_{I,J}$, then $w \in W^J$. Recall also that, if we define $H := I \cap wJw^{-1}$, then each element $x \in W_IwW_J$ has a unique expression of the form $x = uwv$, where $u \in W_I$ is a minimal length coset representative for W_I/W_H, $v \in W_J$ and the following equality is satisfied: $\ell(x) = \ell(u)+\ell(w)+\ell(v)$. In particular, we see that the left W_I orbits in W_IwW_J are parametrized by the minimal length coset representatives W_I/W_H.

From now on, we denote the set of such $u \in W$ by $W^{I,J}$. Note that $W^{I,J}$ is in bijection with the cosets W_Iw/W_J. The proof of our next result is an easy consequence of these observations.

Theorem 5.1. The set of all B orbits in $P_I\backslash G \times G/P_J$ is parametrized by the set

$$D^-_{I,J} := \{(u, w) : \ u \in W^{I,J}, \ w \in X^-_{I,J}\}.$$

Let $x = (u_1, w_1)$ and $y = (u_2, w_2)$ be two elements from $D^-_{I,J}$ and let O_x, O_y denote the corresponding B orbit closures in X. Then

$$\begin{aligned} O_x \subset O_y &\iff u_1w_1 \le u_2w_2 \\ &\iff u_1 \le u_2 \text{ and } w_1 \le w_2. \end{aligned}$$

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