Complex G_2 and Associative Grassmannian

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1 Introduction

An important chapter in differential geometry on holonomy groups is closed by R. Bryant's work in [9], where the long-standing question of existence of metrics with special holonomy groups G_2 and Spin_7 is settled (and it is extended for closed manifolds by D. Joyce). This success story has been a locomotive for many other developments such as deformation theory of G_2-manifolds (see [2]).

The exceptional Lie group G_2, similar to any other Lie group, has different guises depending on the underlying field; it has two real forms and a complex form. All of these incarnations have descriptions as a stabilizer group. We denote these three forms by G_2^+, G_2^-, and by G_2, respectively, where first two are real forms. The first of these is compact, connected, simple, simply connected, of (real) dimension 14, the second group is non-compact, connected, simple, of (real) dimension 14. The complex form of G_2 is connected, simple, simply connected, of (complex) dimension 14.

Let V denote either $\mathbb{R}^7$ or $\mathbb{C}^7$, $e_1, \dots, e_7$ denote its standard basis, and let $x_1 = e_1^*, \dots, x_7 = e_7^*$ denote the dual basis. We write $\mathrm{GL}_7(\mathbb{R})$ or $\mathrm{GL}_7(\mathbb{C})$ instead of $\mathrm{GL}(V)$ when there is no danger of confusion. If the underlying field does not play a role, then we write simply GL_7.

Consider the fourth fundamental representation $\bigwedge^4 V$ of GL_7, which is irreducible. It is peculiar to the numbers $7, 3$ that $\bigwedge^3 V^*$ is naturally isomorphic as a GL_7 representation to $\bigwedge^4 V$, where V^* denotes the dual space. On the other hand, $\bigwedge^4 V$ and $\bigwedge^3 V$ are dual representations, hence $\bigwedge^4 V \simeq (\bigwedge^3 V)^* \simeq \bigwedge^3 V^*$. We use the shorthand e^{ijk} for $x_i \wedge x_j \wedge x_k \in \bigwedge^3 V^*$. Following [9], we set

$$\phi^+ := e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356},$$

and

$$\phi^- := e^{123} - e^{145} - e^{167} - e^{246} + e^{257} + e^{347} + e^{356}.$$

It turns out G_2^- is the stabilizer of ϕ^- in $\mathrm{GL}_7(\mathbb{R})$, G_2^+ is the stabilizer of ϕ^+ in $\mathrm{GL}_7(\mathbb{R})$, and finally, G_2 is the stabilizer of $\phi := \phi^+$ in $\mathrm{GL}_7(\mathbb{C})$. In fact, the orbit $\mathrm{GL}_7 \cdot \phi$ is Zariski open in $\bigwedge^3 V^*$; over real numbers this orbit splits into two with stabilizers G_2^+ and G_2^-. A pleasant consequence of this openness is that $\mathrm{GL}_7(\mathbb{C})$ has only finitely many orbits in $\bigwedge^3 V^*$.

Let $\mathbb{O}_k$ denote the octonion algebra over a field k. To ease our notation, when $k = \mathbb{C}$ we write $\mathbb{O}$. Following [21], we view $\mathbb{O}_k$ as an 8-dimensional composition algebra; it is non-associative, unital (with unity $e \in \mathbb{O}_k$), and it is endowed with a non-degenerate quadratic form[1] $N : \mathbb{O}_k \times \mathbb{O}_k \to k$ such that

$$N(xy) = N(x)N(y) \text{ for all } x, y \in \mathbb{O}_k.$$

Composition algebras exist only in dimensions 1,2,4 and 8. Moreover, they are uniquely determined (up to isotopy) by their quadratic form. When $k = \mathbb{C}$, there is a unique isomorphism class of quadratic forms and any member of this class is *isotropic*, that is to say the norm of the composition algebra vanishes on a nonzero element. When $k = \mathbb{R}$ there are essentially two isomorphism classes of quadratic forms, first of which gives isotropic composition algebras, and the second class gives composition algebras with positive-definite quadratic forms.

It turns out, for any real octonion algebra $\mathbb{O}_\mathbb{R}$,the tensor product $\mathbb{C} \otimes_\mathbb{R} \mathbb{O}_\mathbb{R}$ is isomorphic to $\mathbb{O}$ (by the uniqueness of the octonion algebra over $\mathbb{C}$). In literature $\mathbb{C} \otimes_\mathbb{R} \mathbb{O}_\mathbb{R} = \mathbb{O}$ is known as the "complex bioctonion algebra". We denote by $G_\mathbb{R}$ the group of algebra automorphisms of $\mathbb{O}_\mathbb{R}$, and denote by G the group of algebra automorphisms of $\mathbb{O}$. By Proposition 2.4.6 of [21] we know that the group of $\mathbb{R}$-rational points of G, denoted by $G(\mathbb{R})$, is equal to $G_\mathbb{R}$. Of course(!), $G_\mathbb{R}$ is either G_2^+ or G_2^- depending on which $\mathbb{O}_k$ we start with. Meanwhile, G is equal to G_2. See Theorem 2.3.3, [21].

Let N_1 denote the restriction of the norm N to $e_0^\perp$, the (7 dimensional) orthogonal complement of the identity vector $e_0 \in \mathbb{O}_k$. For notational ease we are going to denote $e_0^\perp$ in $\mathbb{O}_k$ by $\mathbb{I}_k$ and denote $e_0^\perp$ in $\mathbb{O}$ simply by $\mathbb{I}$. The automorphism groups $G_\mathbb{R}$ and G preserve the norms on their respective octonion algebras, and obviously any automorphism maps identity to identity. Thus, we know that $G_\mathbb{R}$ is contained in $SO(\mathbb{I}_k)$ and G is contained in $SO(\mathbb{I})$. Here, $SO(W)$ denotes the group of orthogonal transformations of determinant 1 on a vector space W. If there is no danger of confusion, we write SO_n ($n = \dim W$) in place of $SO(W)$.

A *quaternion algebra*, $\mathbb{D}_k$ over a field k is a 4 dimensional composition algebra over k. As it is mentioned earlier, there are essentially (up to isomorphism) two quaternion algebras over $k = \mathbb{R}$ and there is a unique quaternion algebra over $k = \mathbb{C}$, which we denote simply by $\mathbb{D}$, and call it the split quaternion algebra. (More generally, any composition algebra over $\mathbb{C}$ is called split.) Any split quaternion algebra over $\mathbb{C}$ is isomorphic to the algebra of 2×2 matrices over $\mathbb{C}$ with determinant as its norm.

The split octonion algebra $\mathbb{O}$ over $\mathbb{C}$ has a description which is built on $\mathbb{D}$ by the *Cayley-Dickson doubling process*: As a vector space, $\mathbb{O}$ is equal to $\mathbb{D} \oplus \mathbb{D}$ and its multiplicative structure is given by

$$(a, b)(c, d) = (ac + \bar{d}b, da + b\bar{c}), \quad \text{where } a, b, c, d \in \mathbb{D}, \tag{1}$$

and its norm is defined by $N((a, b)) = N(a) - N(b) = \det a - \det b$.

[1] A quadratic form N on a vector space V over k is a map $N : V \to k$ such that i) $N(\lambda x) = \lambda^2 N(x)$ for all $\lambda \in k$, $x \in V$; ii) The mapping $\langle\, ,\, \rangle : V \times V \to k$ defined by $2\langle x, y\rangle = N(x + y) - N(x) - N(y)$ is bilinear.

Let $\mathrm{Gr}_k(3, \mathbb{I}_k)$ denote the grassmannian of 3 dimensional subspaces in $\mathbb{I}_k$. Let $\mathbb{D}_k \subset \mathbb{O}_k$ be the quaternion subalgebra generated by the first four generators $e_1 = e, e_2, e_3, e_4$ of $\mathbb{O}_k$. As usual, if $k = \mathbb{C}$, then we set $\mathbb{D} = \mathbb{D}_k$. Let us denote by W_0 the intersection $\mathbb{D}_k \cap \mathbb{I}_k$, the span of e_2, e_3, e_4 in $\mathbb{I}_k$. By Corollary 2.2.4 [21], when $k = \mathbb{C}$, we know that G_2 acts transitively on the set of all quoternion subalgebras of $\mathbb{O}$. Since an algebra automorphism fixes the identity, under this action, W_0 is mapped to another 3-plane of the form $W' = D \cap \mathbb{I}$, for some other split quaternion subalgebra $D' \subset \mathbb{O}$. Thus, the G_2-action on $\mathrm{Gr}(3, \mathbb{I})$ has at least two orbits, one of which is $\mathrm{G}_2 \cdot W_0$ and there is at least one other orbit of the form $\mathrm{G}_2 \cdot W$ for some 3-plane W in $\mathbb{I}$. The goal of our paper is to obtain an understanding of the geometry of Zariski closure of the orbit $\mathrm{G}_2 \cdot W_0$. We achieve this goal by using techniques from calibrated geometries.

After we obtained some of our main results we learned from Michel Brion about an earlier work of Alessandro Ruzzi [19, 20]. Since it is closely related to our work, we briefly mention his results. Let G be a connected semisimple group over $\mathbb{C}$, let θ be an involution of G, and let G^θ denote the subgroup of G consisting of elements $g \in G$ that are fixed by θ. Let H be a subgroup of G such that $(G^\theta)^0 \subset H \subset N_G(G^\theta)$. (Here, the superscript 0 indicates the connected component of the identity element.) The quotient varieties of the form G/H are called symmetric varieties. In his 2011 paper, Ruzzi classifies all symmetric varieties of Picard number 1 and in Theorem 2 (a) of 2010 paper [19], he shows that the smooth equivariant completion with Picard number one of the symmetric variety $\mathrm{G}_2/\mathrm{SL}_2 \times \mathrm{SL}_2$, which we denote by X_{min}, is the intersection of the grassmannian $\mathrm{Gr}(3, \mathbb{I})$ with a 27 dimensional G_2-stable linear space in $\mathbb{P}(\bigwedge^3 \mathbb{I})$. (Over $\mathbb{C}$, $\mathrm{SL}_2 \times \mathrm{SL}_2$ is identified with the special orthogonal group SO_4.)

Here in our paper, we analyze the natural action of the maximal torus of G_2 on X_{min}. In particular, we compute the Poincaré polynomial of X_{min} by using the Białynicki-Birula decomposition. We should mention that, over the field of real numbers, the symmetric variety $\mathrm{ASS} := \mathrm{G}_2^+/\mathrm{SO}_4(\mathbb{R})$ is already compact, and its geometry is well understood; a complete description of the ring structure of the $H^*(\mathrm{ASS}, \mathbb{Z})$ is described in [4].

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2 Grassmann of 3-planes

For a subset $S \subset \mathbb{O}_k$, we denote by $A(S)$ the subalgebra of $\mathbb{O}_k$ that is generated by S. Let $W \in \mathrm{Gr}_k(3, \mathbb{I}_k)$ be a 3-plane and let $u_1, u_2, u_3 \in W$ be a basis. Thus, the vector space dimension of $A(W)$ is either $\dim A(W) = 4$ or $\dim A(W) = 8$. In latter case, obviously, $A(W) = \mathbb{O}_k$. In the former case, $A(W)$ is an associative subalgebra (as follows from Proposition 1.5.2 of [21]) but it does not need to be a composition subalgebra. Our orbit $G_k \cdot W_0$ in $\mathrm{Gr}_k(3, \mathbb{I}_k)$ is precisely that set of 3-planes, where $A(W)$ is a 4 dimensional composition subalgebra, which we state in our next lemma:

Lemma 2.1. If $W \subset \mathbb{I}$ be a 3-plane that is in the G_2-orbit of W_0, then $A(W)$ is a quaternion subalgebra of $\mathbb{O}$. Moreover, the stabilizer subgroup of any such W is isomorphic to SO_4, the special orthogonal group of 4 by 4 matrices.

Proof. G_2 acts transitively on the set of quaternion subalgebras (Corollary 2.2.4 [21]), hence our first assertion follows. To prove our second claim it is enough to prove it for W_0, the "origin of the orbit" since other stabilizers are isomorphic to each other by conjugation.

Observe that $g \in G$ fixes W_0 if and only if it fixes $\mathbb{D}$ if and only if it fixes the orthogonal complement of $\mathbb{D}$ in $\mathbb{O}$, which is a 4 dimensional vector space. Therefore, on one hand we have an injection $\iota: \ \mathrm{Stab}_G(W_0) \hookrightarrow SO(\mathbb{D}^{\perp}) \simeq \mathrm{SO}_4$. On the other hand, we know that the elements of $SO(\mathbb{D}^{\perp})$ are completely determined by how they act on the part of the basis e_5, e_6, e_7, e_8 of $\mathbb{O}$. Indeed, we see this from Figure 1, which gives us the multiplicative structure of $\mathbb{O}$.

	e	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e	e	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_2	e_2	$-e$	e_4	$-e_3$	e_6	$-e_5$	$-e_8$	e_7
e_3	e_3	$-e_4$	$-e$	e_2	e_7	e_8	$-e_5$	$-e_6$
e_4	e_4	$-e_3$	$-e_2$	$-e$	e_8	$-e_7$	e_6	$-e_5$
e_5	e_5	$-e_6$	$-e_7$	$-e_8$	e	$-e_2$	$-e_3$	$-e_4$
e_6	e_6	e_5	$-e_8$	e_7	e_2	e	e_4	$-e_3$
e_7	e_7	e_8	e_5	$-e_6$	e_3	$-e_4$	e	e_2
e_8	e_8	$-e_7$	e_6	e_5	e_4	e_3	$-e_2$	e

Figure 1: Multiplication table for split octonions.

The multiplication table of e_5, e_6, e_7, e_8 includes e_2, e_3, e_4 and e, therefore, the action of g on W is uniquely determined by the action of g on e_5, e_6, e_7, e_8. It follows that ι is surjective as well, hence it is an isomorphism. □

Remark 2.2. The element-wise stabilizer of $\mathbb{D}$ in G is isomorphic to SL_2. (See Proposition 2.2.1 [21]). Heuristically, this follows from the fact that $\mathbb{O} = \mathbb{D} \oplus \mathbb{D}$, and that $(\mathbb{D}, N) = (\mathrm{Mat}_2, \det)$.

Since $\mathbb{O} = \mathbb{D} \oplus \mathbb{D}$ and $\mathbb{D} = \mathrm{Mat}_2$, we take $\{(e_{ij}, 0) : \ i, j = 1, 2\} \cup \{(0, e_{ij}) : \ i, j = 1, 2\}$ as a basis for $\mathbb{O}$. Here, e_{ij} is the 2×2 matrix with 1 at the i, jth position and 0's everywhere else. Recall that $\mathbb{I}$ is the orthogonal complement of the identity $e = (e_{11} + e_{22}, 0)$ of $\mathbb{O}$. A straightforward computation shows that $(x, y) \in \mathbb{O}$ is in $\mathbb{I}$ if and only if the trace of x is 0. Thus, we write $\mathbb{I} = \mathfrak{sl}_2 \oplus \mathrm{Mat}_2$ (we are going to make use of Lie algebra structure on $\mathfrak{sl}_2$ in the sequel).

Let $W \in \mathrm{Gr}(3, \mathbb{I})$ be a 3-plane in $\mathbb{I}$ and let $\{u_1, u_2, u_3\}$ be a basis for W. The map $P : \mathrm{Gr}(3, \mathbb{I}) \to \mathbb{P}(\bigwedge^3 \mathbb{I})$ defined by $P(W) = [u_1 \wedge u_2 \wedge u_3]$ is the Plücker embedding of

$\mathrm{Gr}(3,\mathbb{I})$ into 34 dimensional projective space $\mathbb{P}(\bigwedge^3 \mathbb{I})$. Note that $\mathrm{GL}(\mathbb{I})$ acts on both of the varieties $\mathrm{Gr}(3,\mathbb{I})$ and $\bigwedge^3 \mathbb{I}$ via its natural action on $\mathbb{I}$. Note also that the Plücker embedding is equivariant with respect to these actions. In particular, it is equivariant with respect to the subgroup G_2.

We make the identifications

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{i} \leftrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ \mathbf{j} \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \mathbf{k} \leftrightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{2}$$

and take $\{(\mathbf{i},0),(\mathbf{j},0),(\mathbf{k},0),(0,1),(0,\mathbf{i}),(0,\mathbf{j}),(0,\mathbf{k})\}$ as a basis for $\mathbb{I}$. Let W_0 denote the span of $\{(\mathbf{i},0),(\mathbf{j},0),(\mathbf{k},0)\}$ and let W_0^* denote the span of $\{(0,\mathbf{i}),(0,\mathbf{j}),(0,\mathbf{k})\}$. Thus,

$$\mathbb{I} = W_0 \oplus W_0^* \oplus \mathbb{C}. \tag{3}$$

Remark 2.3. It is noted earlier that a copy of $\mathfrak{sl}_2$ sits in $\mathbb{I}$:

$$\mathfrak{sl}_2 = \mathfrak{sl}_2 \oplus 0 \hookrightarrow \mathfrak{sl}_2 \oplus \mathrm{Mat}_2 = \mathbb{I}.$$

This copy of $\mathfrak{sl}_2$ is W_0 as a vector space.

Remark 2.4. A straightforward calculation shows that if $(0,v) \in W_0^*$, then for all $(x,0) \in \mathfrak{sl}_2$, $(x,0)(0,v) = (0,vx)$.

Next, we elaborate on a portion of the discussion from [14], §22.3 and analyze $\bigwedge^3 \mathbb{I}$ more closely.

Let U denote $W_0 \oplus W_0^*$ so that we have $\bigwedge^3(W_0 \oplus W_0^* \oplus \mathbb{C}) = \bigoplus_{n=0}^3 \bigwedge^n U \otimes \bigwedge^{3-n} \mathbb{C} = \bigwedge^3 U \oplus \bigwedge^2 U$. Since $\dim W_0 = \dim W_0^* = 3$, we have canonical identifications $W_0 = \bigwedge^2 W_0^*$ and $W_0^* = \bigwedge^2 W_0$. It follows that

$$\begin{aligned}\bigwedge^3 U &= (\mathbb{C}\otimes\mathbb{C}) \oplus (W_0 \otimes \bigwedge^2 W_0^*) \oplus (\bigwedge^2 W_0 \otimes W_0^*) \oplus (\mathbb{C}\otimes\mathbb{C}) \\ &= \mathbb{C} \oplus (W_0 \otimes W_0) \oplus (W_0^* \otimes W_0^*) \oplus \mathbb{C}\end{aligned}$$

and that

$$\begin{aligned}\bigwedge^2 U &= \bigwedge^2 W_0 \otimes \mathbb{C} \oplus W_0 \otimes W_0^* \oplus \mathbb{C} \otimes \bigwedge^2 W_0^* \\ &= W_0^* \oplus (W_0 \otimes W_0^*) \oplus W_0.\end{aligned}$$

Putting all of the above together we see that

$$\begin{aligned}\bigwedge^3 \mathbb{I} &= \mathbb{C} \oplus (W_0 \otimes W_0) \oplus (W_0^* \otimes W_0^*) \oplus \mathbb{C} \oplus W_0^* \oplus (W_0 \otimes W_0^*) \oplus W_0 \\ &= \mathbb{I} \oplus (W_0 \otimes W_0) \oplus (W_0^* \otimes W_0^*) \oplus (W_0 \otimes W_0^*) \oplus \mathbb{C}.\end{aligned}$$

Next, we analyze $\mathrm{Sym}^2\mathbb{I}$ more closely;

$$\begin{aligned}\mathrm{Sym}^2\mathbb{I} &= \mathrm{Sym}^2(U\oplus\mathbb{C})\\ &= (\mathrm{Sym}^2U\otimes\mathbb{C})\oplus(\mathrm{Sym}^1U\otimes\mathbb{C})\oplus(\mathbb{C}\otimes\mathrm{Sym}^2\mathbb{C})\\ &= \mathrm{Sym}^2W_0\oplus(W_0\otimes W_0^*)\oplus\mathrm{Sym}^2W_0^*\oplus(W_0\oplus W_0^*)\oplus\mathbb{C}\\ &= (\mathrm{Sym}^2W_0\oplus W_0^*)\oplus(W_0\otimes W_0^*)\oplus(\mathrm{Sym}^2W_0^*\oplus W_0)\oplus\mathbb{C}\\ &= (\mathrm{Sym}^2W_0\oplus\bigwedge^2 W_0)\oplus(W_0\otimes W_0^*)\oplus(\mathrm{Sym}^2W_0^*\oplus\bigwedge^2 W_0^*)\oplus\mathbb{C}\\ &= \mathrm{End}(W_0)\oplus(W_0\otimes W_0^*)\oplus\mathrm{End}(W_0^*)\oplus\mathbb{C}\\ &\simeq (W_0\otimes W_0)\oplus(W_0\otimes W_0^*)\oplus(W_0^*\otimes W_0^*)\oplus\mathbb{C}.\end{aligned}$$

Remark 2.5. The last term is only an isomorphism since we are using non-canonical identification of W_0 with W_0^*.

Therefore, we see that

$$\bigwedge^3\mathbb{I}\simeq\mathrm{Sym}^2\mathbb{I}\oplus\mathbb{I}. \tag{4}$$

Furthermore, it is true that $\mathrm{Sym}^2\mathbb{I}=\Gamma_{2,0}\oplus\mathbb{C}$, where

$$\begin{aligned}\Gamma_{2,0} &= (W_0\otimes W_0)\oplus(W_0^*\otimes W_0^*)\oplus(W_0\otimes W_0^*)\\ &= (W_0\otimes\bigwedge^2 W_0^*)\oplus(\bigwedge^2 W_0\otimes W_0^*)\oplus(W_0\otimes W_0^*).\end{aligned} \tag{5}$$

is an irreducible representation of G_2 with highest weight $2\omega_1$, where ω_1 is the highest weight of the first fundamental representation $\mathbb{I}$ of G_2. (See [14], §22.3.) Once the root system $\Phi=\{\alpha_1,\dots,\alpha_6,\beta_1,\dots,\beta_6\}$ is chosen as in [14], §22.2 (pg. 347), we see that $2\omega_1=\alpha_1+\alpha_3+\alpha_4$.

The structure of the representation of G_2 on $\mathbb{I}$ can be spelled out to a finer degree once we linearize the action. Let $\mathfrak{g}_2$ denote the Lie algebra of G_2. It is well known that $\mathfrak{g}_2$ contains a copy of $\mathfrak{g}_0=\mathfrak{sl}_3$, and moreover, as a representation of $\mathfrak{g}_0$ it has the following decomposition:

$$\mathfrak{g}_2=\mathfrak{g}_0\oplus W\oplus W^*,$$

where W is a suitable 3 dimensional subspace. Furthermore, the unique 7 dimensional irreducible representation V of $\mathfrak{g}_2$ can be identified with $V=W\oplus W^*\oplus\mathbb{C}$ as an $\mathfrak{sl}_3$-module. In our notation, we are going to take W as W_0. Before making this identification we choose a basis for W using roots system $\Phi=\{\alpha_1,\dots,\beta_{12}\}$.

Let $V_i\subset V$ $(i=1,\dots,6)$ denote the eigenspace (corresponding to the eigenvalue α_i) for the action of the maximal abelian subalgebra $\mathfrak{h}\subset\mathfrak{g}_2$ corresponding to Φ. Let Y_i be the root vector whose eigenvalue is $-\alpha_i=\beta_i$ for $i=1,\dots,6$. Arguing as in pg. 354 of [14], we have the basis $e_1 = v_4, e_2 = w_1, e_3 = w_3$ for W, where w_i's are found as follows:

$$\begin{aligned} v_3 &= Y_1(v_4), \\ v_1 &= -Y_2(v_3), \\ u &= Y_1(v_1), \\ w_1 &= \frac{1}{2}Y_1(u), \\ w_3 &= Y_2(w_1), \\ w_4 &= -Y_1(w_3). \end{aligned}$$

The corresponding dual basis elements e_1^*, e_2^*, e_3^* are given by w_4, v_1, v_3, respectively. The upshot of all of these is that we identify W_0 with W in such a way that the basis v_4, w_1, w_3 corresponds (in the given order) to $(\mathbf{i}, 0), (\mathbf{j}, 0), (\mathbf{k}, 0)$, and the basis w_4, v_1, v_3 for W^* corresponds to $(0, \mathbf{i}), (0, \mathbf{j}), (0, \mathbf{k})$. With respect to these identifications, we observe that the highest weight vector in $\mathrm{Sym}^2 V \subset \bigwedge^3 V$ of the highest weight $2\omega_1 = \alpha_1 + \alpha_3 + \alpha_4$ is given by the 3-form $v_1 \wedge v_3 \wedge v_4$, or by $(\mathbf{i}, 0) \wedge (0, \mathbf{j}) \wedge (0, \mathbf{k})$ in the case of $\mathrm{Sym}^2 \mathbb{I} \subset \bigwedge^3 \mathbb{I}$. It is clear that the 3-form $(\mathbf{i}, 0) \wedge (0, \mathbf{j}) \wedge (0, \mathbf{k})$ is actually an element of $W_0 \otimes \bigwedge^2 W_0^* \subset \Gamma_{2,0}$ by (5).

It is straightforward to verify that the octonions $(\mathbf{i}, 0), (0, \mathbf{j})$, and $(0, \mathbf{k})$ generate a quaternion algebra which we denote by $\mathbb{U}$. By Lemma 2.1 we see that the stabilizer subgroup of $\mathbb{U}$ is SO_4. It is well known that the highest weight vector in $\Gamma_{2,0} \subset \mathrm{Sym}^2(\mathbb{I})$ is the direction vector of the line that is stabilized by SO_4.

We view the projectivization $\mathbb{P}(\Gamma_{2,0})$ as a (closed) subvariety of $\mathbb{P}(\bigwedge^3 \mathbb{I})$. The image of $\mathrm{Gr}(3, \mathbb{I})$ intersects $\mathbb{P}(\Gamma_{2,0})$. In fact, by the above discussion we know that the image of the 3-plane $U_0 := \mathbb{U} \cap \mathbb{I} \in \mathrm{Gr}(3, \mathbb{I})$ under Plücker embedding is the SO_4-fixed point $[u_0] \in \mathbb{P}(\Gamma_{2,0})$. On one hand, since it is a G_2-equivariant isomorphism onto its image, the orbit $\mathrm{G}_2 \cdot U_0$ in $\mathrm{Gr}(3, \mathbb{I})$ is mapped isomorphically onto $\mathrm{G}_2 \cdot [u_0]$ in $\mathbb{P}(\Gamma_{2,0}) \subset \mathbb{P}(\bigwedge^3 \mathbb{I})$. On the other hand, as we are going to see in the sequel, the closure of the orbit $\mathrm{G}_2 \cdot U_0$ in $\mathrm{Gr}(3, \mathbb{I})$ is smooth, however, the Zariski closure of the orbit $\mathrm{G}_2 \cdot [u_0]$ in $\mathbb{P}(\bigwedge^3 \mathbb{I})$ is not. The latter closure is the smallest "degenerate" compactification of the symmetric variety $\mathrm{G}_2/\mathrm{SO}_4$, whereas the former compactification is the smallest, smooth G_2-equivariant compactification.

3 More on Octonions

In this section we collect and improve some known facts about alternating forms on (split) composition algebras.

The multiplicative structure of a quaternion algebra (over $\mathbb{R}$ or $\mathbb{C}$) is always associative (but not commutative). To measure how badly the associativity of multiplication fails in $\mathbb{O}$ one looks at the *associator*, defined by

$$[x, y, z] = (xy)z - x(yz) \text{ for all } x, y, z \in \mathbb{O}_k. \tag{6}$$

It is well known that the associator is an alternating 3-form (see Section 1.4 of [21]).

There are several other related multiplication laws on the imaginary part $\mathbb{I}_k$ of $\mathbb{O}_k$. For example, the "cross-product" is defined by

$$a \times b = \frac{1}{2}(ab - ba) \text{ for all } a, b \in \mathbb{I}_k.$$

Obviously, the cross-product is alternating.

The "dot product" is defined by

$$a \cdot b = -\frac{1}{2}(ab + ba) \text{ for all } a, b \in \mathbb{I}_k.$$

It is also obvious that $ab = a \times b - a \cdot b$ for $a, b \in \mathbb{I}_k$.

These products are easily extended to $\mathbb{O}_k$. Indeed, any element of $\mathbb{O}_k$ has the form $x = \alpha(1, 0) + a$, where $\alpha \in k$, $a \in \mathbb{I}_k$, and if $y = \beta(1, 0) + b$ is another element from $\mathbb{O}_k$, then

$$xy = (\alpha, a)(\beta, b) = (\alpha\beta - a \cdot b, \alpha a + \beta b + a \times b), \tag{7}$$

where we use the identification $\alpha(1, 0) + a = (\alpha, a)$. In particular, if $x = a$ and $y = b$ are from $\mathbb{I}_k$, then $xy = (-x \cdot y, x \times y)$, hence

$$N(xy) = (x \cdot y)^2 + N(x \times y). \tag{8}$$

Now we focus on $k = \mathbb{C}$ and extend some results from [16] to our setting. First, we re-label the basis $\{e, e_1, \ldots, e_7\}$ for $\mathbb{O}$ so that $\{e_1 = (\mathbf{i}, 0), e_2 = (\mathbf{j}, 0), e_3 = (\mathbf{k}, 0), e_4 = (0, 1), e_5 = (0, \mathbf{i}), e_6 = (0, \mathbf{j}), e_7 = (0, \mathbf{k})\}$ is the standard basis for $\mathbb{I}$. Consider the trilinear form

$$\varphi(x, y, z) = \langle x \times y, z\rangle, \qquad x, y, z \in \mathbb{I}.$$

Lemma 3.1. φ is an alternating 3-form on $\mathbb{I}$.

Proof. Since both cross-product and the inner product $\langle , \rangle$ are bilinear, we see it is enough to check the assertion on the basis $\{e_1, \ldots, e_7\}$. We verified this by using software called Maple. □

Remark 3.2. It follows from (7) that

$$\varphi(x, y, z) = \langle xy, z\rangle \;\text{ for } x, y, z \in \mathbb{I}. \tag{9}$$

It is not difficult to verify (by using Maple, or by hand) that φ is equal to the 3-form

$$\varphi = e^{123} - e^{145} + e^{167} - e^{246} - e^{257} - e^{347} + e^{356}, \tag{10}$$

where $e^{ijk} = de_i \wedge de_j \wedge de_k$ as before. In particular, we see that from (9) that $\mathrm{G}_2 = \mathrm{Aut}(\mathbb{O})$ stabilizes the form (10).

Definition 3.3. For a 3-plane $W \in \mathrm{Gr}(3, \mathbb{I})$ we define $\varphi(W)$ to be the evaluation of φ on any orthonormal basis $\{x, y, z\}$ of W. We call a 3-plane $W \in \mathrm{Gr}(3, \mathbb{I})$ associative if $W = D \cap \mathbb{I}$, where D is a (necessarily split) quaternion subalgebra of $\mathbb{O}$.

Theorem 3.4. A 3-plane $W \in \mathrm{Gr}(3, \mathbb{I})$ is associative if and only if $\varphi(x, y, z) \in \{-1, +1\}$ for any orthonormal basis $\{x, y, z\}$ of W.

Proof. Any two elements x, y of an orthonormal triplet (x, y, z) from $\mathbb{I}$ form a "special $(1,1)$-pair" in the sense of [21], Definition 1.7.4.[2] Since G_2 acts transitively on special $(1,1)$-pairs, and since $((\mathbf{i}, 0), (\mathbf{j}, 0))$ is such, there exists $g \in \mathrm{G}_2$ such that $g(x) = (\mathbf{i}, 0)$, $g(y) = (\mathbf{j}, 0)$. In particular, it follows that $g(xy) = (\mathbf{k}, 0)$.

We claim that x, y, z generates a quaternion algebra if and only if $g(z) = \pm(\mathbf{k}, 0)$. Indeed, unless xy is a scalar multiple of z, the span in $\mathbb{I}$ of e, x, y, z and xy is 5 dimensional, hence the composition algebra generated by x, y, z is not a quaternion subalgebra. It follows that $\{1, x, y, z\}$ is an orthonormal basis for a quaternion subalgebra if and only if z is a scalar multiple of xy. Since the norm of z is 1, we see that $z = \pm xy$, hence $g(z) = \pm(\mathbf{k}, 0)$. Then

$$\begin{aligned}
\langle xy, z\rangle &= \langle g^{-1}((\mathbf{i},0))g^{-1}((\mathbf{j},0)), g^{-1}(\pm(\mathbf{k},0))\rangle \\
&= \langle g^{-1}((\mathbf{i},0)(\mathbf{j},0)), g^{-1}(\pm(\mathbf{k},0))\rangle \\
&= \langle (\mathbf{i},0)(\mathbf{j},0), \pm(\mathbf{k},0)\rangle \\
&= \pm 1.
\end{aligned}$$

□

We define the *triple-cross product* on $\mathbb{O}$ as follows

$$x \times y \times z = \frac{1}{2}(x(\bar{y}z) - z(\bar{y}x)) \quad \text{for all } x, y, z \in \mathbb{O}. \tag{11}$$

Lemma 3.5. The triple-cross product is trilinear and alternating. Moreover, $N(x \times y \times z) = N(x)N(y)N(z)$ for all $x, y, z \in \mathbb{O}$.

Proof. The trilinearity is obvious. To prove the second claim we check $x \times x \times z = 0$, $x \times y \times y = 0$, and $x \times y \times x = 0$. We use [21], Lemma 1.3.3 *i*) for the first two:

$$\begin{aligned}
x \times x \times z &= \frac{1}{2}(x(\bar{x}z) - z(\bar{x}x)) = \frac{1}{2}(N(x)z - zN(x)) = 0, \\
x \times y \times y &= \frac{1}{2}(x(\bar{y}y) - y(\bar{y}x)) = \frac{1}{2}(xN(y) - N(y)x) = 0, \\
x \times y \times x &= \frac{1}{2}(x(\bar{y}x) - x(\bar{y}x)) = 0.
\end{aligned}$$

Finally, to prove $N(x \times y \times z) = N(x)N(y)N(z)$ we expand $x \times y \times z$ in the orthonormal basis $e = (1, 0), e_1, \ldots, e_7$ of $\mathbb{O}$. In particular, this allows us to assume that x, y and z as scalar multiples of standard basis vectors. Now it is straightforward to verify (by Maple) that $x(\bar{y}z) = -z(\bar{y}x)$, hence $x \times y \times z = x(\bar{y}z)$ and our claim follows by taking norms. □

[2] A pair (x, y) of elements from a composition algebra is called a special (λ, μ)-pair if $\langle x, e\rangle = \langle y, e\rangle = \langle x, y\rangle = 0$, $N(x) = \lambda$ and $N(y) = \mu$.

Next, we show that the "associator identity" of Harvey-Lawson (Theorem 1.6 [16]) holds for split octonion algebras.

Theorem 3.6. For all $x, y, z \in \mathbb{I}$, the associator $[x, y, z]$ lies in $\mathbb{I}$, and

$$x \times y \times z = \langle xy, z\rangle e + [x, y, z], \tag{12}$$

where e is the identity element $(1, 0)$ of $\mathbb{O}$. Moreover,

$$|x \wedge y \wedge z| = \varphi(x, y, z)^2 + N([x, y, z]), \tag{13}$$

where $|x \wedge y \wedge z|$ is defined as $N(x)N(y)N(z)$.

Proof. By linearity and alternating property, it is enough to prove our first two assertions for (orthonormal) triplets (x, y, z) from the basis $\{e_1, \dots, e_7\}$ and once again this is straightforward to verify by using Maple. To prove (13), we note as in the proof of Lemma 3.5 that $N(x \times y \times z) = N(x(\bar{y}z)) = N(x)N(y)N(z) = |x \wedge y \wedge z|$. Thus, our claim now follows from the simple fact that $N(\alpha e + u) = \alpha^2 + N(u)$ whenever $u \in e^{\perp}, \alpha \in \mathbb{C}$. □

Corollary 3.7. Let $W \in \mathrm{Gr}(3, \mathbb{I})$ be a 3-plane and let x, y, z be an orthonormal basis for W (which always exists by Gram-Schmidt process). Then W is associative if and only if $[x, y, z] = 0$.

Proof. By linearity and alternating property, it is enough to prove the statement "$\varphi(x, y, z) \in \{-1, +1\}$ if and only if $[x, y, z] = 0$" on the orthonormal basis $\{e_1, \dots, e_7\}$. We verified the cases by using Maple. □

Recall that when $x, y,$ and z are orthonormal vectors from $\mathbb{I}$, $x \times y \times z = x(\bar{y}z)$. We verified by using Maple that if the associator $[x, y, z]$ is nonzero for $\{x, y, z\} \subset \{e_1, \dots, e_7\}$, then

$$x \times y \times z = [x, y, z] \quad \text{and} \quad \varphi(x, y, z) = 0, \tag{14}$$

and if $[x, y, z] = 0$, then

$$x \times y \times z = \varphi(x, y, z)e, \tag{15}$$

which conforms with (12). It is not difficult to show that the form $*\varphi(u, v, w, z)$ defined by

$$*\varphi(u, v, w, z) = \langle u \times v \times w, z\rangle$$

is an alternating 4-form, and it can be expressed as in

$$*\varphi = -e^{4567} + e^{2367} - e^{2345} + e^{1357} + e^{1346} + e^{1256} - e^{1247}. \tag{16}$$

This can be seen from the fact that the monomials of φ and $*\varphi$ are complementary in the sense that e^{ijkl} appears in $*\varphi$ if any only if there exists unique monomial e^{rst} in φ such that $\{r, s, t, i, j, k, l\} = \{1, 2, 3, 4, 5, 6, 7\}$. Also, it can be checked directly on the standard orthogonal basis. Now, we define a new 3-form $\chi(u, v, w)$ by the identity $\langle \chi(u, v, w), z\rangle = *\varphi(u, v, w, z)$.

An important consequence of these definitions and the above discussion (specifically, the equation (14)) is that if a 3-plane spanned by $x, y, z \in \mathbb{I}$ generates a quaternion subalgebra, then $[x, y, z] = 0$, which implies $\chi(x, y, z) = 0$. We record this in our next lemma:

Lemma 3.8. If the 3-plane W generated by $x, y, z \in \mathbb{I}$ is associative, then $\chi(x, y, z) = 0$.

The vanishing locus of χ on $\mathbb{P}(\bigwedge^3 \mathbb{I})$ can be made more precise since

$$\begin{aligned}\chi = &(e^{247} - e^{256} - e^{346} - e^{357})e_1\\ &+ (e^{156} - e^{147} + e^{345} - e^{367})e_2\\ &+ (-e^{245} + e^{267} + e^{146} + e^{157})e_3\\ &+ (e^{567} + e^{127} - e^{136} + e^{235})e_4\\ &+ (-e^{126} - e^{467} - e^{137} - e^{234})e_5\\ &+ (e^{457} + e^{125} + e^{134} - e^{237})e_6\\ &+ (e^{135} - e^{124} - e^{456} + e^{236})e_7,\end{aligned}$$

which follows from (16).

Remark 3.9. It appears that the idea of using these seven linear equations obtained from χ to study associative manifolds is first used in [3].

4 Charts of Gr$(3, \mathbb{I})$

We recall some standard facts. Let p_I denote a variable indexed by a sequence $I = i_1 i_2 \dots i_d$ of elements from $\{1, \dots, n\}$. We assume that if the order of two entries of the sequence I are interchanged, hence we have another sequence I' of the same length, then $p_I = -p_{I'}$. In particular, it follows that if two entries of I are equal, then $p_I = 0$. We are going to view p_I's as coordinate functions on $\mathbb{P}(\bigwedge^d \mathbb{C}^n)$.

The homogenous coordinate ring of the Grassmann variety of d dimensional subspaces in $\mathbb{C}^n$ is the quotient of the polynomial ring $\mathbb{C}[p_I : \ I \subset \{1, \dots, n\},\ |I| = d]$ by the ideal generated by the following quadratic polynomials: $\sum_{s=1}^{d+1} (-1)^s p_{i_1 i_2 \dots i_{d-1} j_s} p_{j_1 j_2 \dots \widehat{j_s} \dots j_{d+1}}$, where $i_1, \dots, i_{d-1}, j_1, \dots, j_{d+1}$ are arbitrary numbers from $\{1, \dots, n\}$. Here, the hatted entry $\widehat{j_s}$ is omitted from the sequence. Of course, the case of interest for us is when $d = 3$, $n = 7$, and (17) is a (at most) 4-term relation:

$$p_{i_1 i_2 j_1} p_{j_2 j_3 j_4} = p_{i_1 i_2 j_2} p_{j_1 j_3 j_4} - p_{i_1 i_2 j_3} p_{j_1 j_2 j_4} + p_{i_1 i_2 j_4} p_{j_1 j_2 j_3}. \tag{17}$$

To see the complex manifold structure on $\mathrm{Gr}(d, \mathbb{C}^n)$, one looks at the intersections of $\mathrm{Gr}(d, \mathbb{C}^n)$ with the standard open charts in $\mathbb{P}(\wedge^d \mathbb{C}^n)$:

$$U_I := \mathrm{Gr}(d, \mathbb{C}^n) \cap \{x \in \mathbb{P}(\bigwedge^d \mathbb{C}^n) : \ p_I(x) \neq 0\}.$$

It is not difficult to show that the coordinate functions on U_I are given by p_J/p_I, where $J = j_1 \dots j_d$ is a sequence satisfying $|\{j_1, \dots, j_d\} \cap \{i_1, \dots, i_d\}| = d-1$. Indeed, it is not difficult to verify (by using Plücker relations) that any other rational function of the form p_K/p_I is a polynomial in p_J/p_I's. These are all well known facts. See for example [17].

We compute some examples in our case of interest. Let $J = 123$ and $I = 456$. By (17) we compute $p_I p_J$ and divide the result by p_J^2:

$$\frac{p_{456}}{p_{123}} = \frac{p_{145}}{p_{123}}\frac{p_{236}}{p_{123}} + \frac{p_{245}}{p_{123}}\frac{p_{136}}{p_{123}} + \frac{p_{345}}{p_{123}}\frac{p_{126}}{p_{123}}. \tag{18}$$

Now, to ease our notation, we denote the coordinate functions p_{abc}/p_{123} on U_{123} by q_{abc}. Our goal is to describe each q_{abc} by using $\{q_{124}, q_{125}, q_{126}, q_{127}, q_{134}, q_{135}, q_{136}, q_{137}, q_{234}, q_{235}, q_{236}, q_{237}\}$, only. We have the following quadratic identities

$$\begin{aligned}
q_{145} &= -q_{124}q_{135} - q_{134}q_{125}\\
q_{146} &= -q_{124}q_{136} - q_{134}q_{126}\\
q_{147} &= -q_{124}q_{137} - q_{134}q_{127}\\
q_{156} &= -q_{125}q_{136} - q_{135}q_{126}\\
q_{157} &= -q_{125}q_{137} - q_{135}q_{127}\\
q_{167} &= -q_{126}q_{137} - q_{136}q_{127}
\end{aligned}$$

$$\begin{aligned}
q_{245} &= q_{124}q_{235} - q_{234}q_{125}\\
q_{246} &= q_{124}q_{236} - q_{234}q_{126}\\
q_{247} &= q_{124}q_{237} - q_{234}q_{127}\\
q_{256} &= q_{125}q_{236} - q_{235}q_{126}\\
q_{257} &= q_{125}q_{237} - q_{235}q_{127}\\
q_{267} &= q_{126}q_{237} - q_{236}q_{127}
\end{aligned}$$

$$\begin{aligned}
q_{345} &= q_{134}q_{235} + q_{234}q_{135}\\
q_{346} &= q_{134}q_{236} + q_{234}q_{136}\\
q_{347} &= q_{134}q_{237} + q_{234}q_{137}\\
q_{356} &= q_{135}q_{236} + q_{235}q_{136}\\
q_{357} &= q_{135}q_{237} + q_{235}q_{137}\\
q_{367} &= q_{136}q_{237} + q_{236}q_{137}.
\end{aligned}$$

Thus, we obtain cubic identities

$$\begin{aligned}
q_{456} &= (-q_{124}q_{135} - q_{134}q_{125})q_{236} + (q_{124}q_{235} - q_{234}q_{125})q_{136} + (q_{134}q_{235} + q_{234}q_{135})q_{126}\\
q_{457} &= (-q_{124}q_{135} - q_{134}q_{125})q_{237} + (q_{124}q_{235} - q_{234}q_{125})q_{137} + (q_{134}q_{235} + q_{234}q_{135})q_{127}\\
q_{467} &= (-q_{124}q_{136} - q_{134}q_{126})q_{237} + (q_{124}q_{236} - q_{234}q_{126})q_{137} + (q_{134}q_{236} + q_{234}q_{136})q_{127}\\
q_{567} &= (-q_{125}q_{136} - q_{135}q_{126})q_{237} + (q_{125}q_{236} - q_{235}q_{126})q_{137} + (q_{135}q_{236} + q_{235}q_{136})q_{127}.
\end{aligned}$$

In the previous section we expressed the vanishing locus of χ on $\mathbb{P}(\bigwedge^3 \mathbb{I})$ in terms of orthogonal frames, which in turn can be expressed in Plücker coordinates by the following 7 linear equations:

$$p_{247} - p_{256} - p_{346} - p_{357} = 0 \tag{19}$$
$$p_{156} - p_{147} + p_{345} - p_{367} = 0 \tag{20}$$
$$-p_{245} + p_{267} + p_{146} + p_{157} = 0 \tag{21}$$
$$p_{567} + p_{127} - p_{136} + p_{235} = 0 \tag{22}$$
$$-p_{126} - p_{467} - p_{137} - p_{234} = 0 \tag{23}$$
$$p_{457} + p_{125} + p_{134} - p_{237} = 0 \tag{24}$$
$$p_{135} - p_{124} - p_{456} + p_{236} = 0. \tag{25}$$

Moreover, it follows from Lemma 3.8 that the variety of all associative 3-planes in $\mathbb{P}(\bigwedge^3 \mathbb{I})$, namely the image of the associative grassmannian $\mathrm{G}_2/\mathrm{SO}_4$ under the Plücker embedding of $\mathrm{Gr}(3, \mathbb{I})$ lies in the intersection of these 7 hyperplanes with $\mathrm{Gr}(3, \mathbb{I})$. We express these 7 defining linear forms in terms of the coordinates on U_{123} (in the above order) and they become quadratic and cubic polynomials:

$$\begin{aligned} f_1 := {} & q_{124}q_{237} - q_{236}q_{125} + q_{126}q_{235} - q_{127}q_{234} \\ & - q_{134}q_{236} - q_{135}q_{237} - q_{136}q_{234} - q_{137}q_{235} \end{aligned} \tag{26}$$
$$\begin{aligned} f_2 := {} & q_{124}q_{137} - q_{125}q_{136} - q_{135}q_{126} + q_{134}q_{127} \\ & + q_{134}q_{235} + q_{234}q_{135} - q_{136}q_{237} - q_{236}q_{137} \end{aligned} \tag{27}$$
$$\begin{aligned} f_3 := {} & -q_{124}q_{136} - q_{124}q_{235} - q_{125}q_{137} + q_{234}q_{125} \\ & - q_{134}q_{126} + q_{126}q_{237} - q_{135}q_{127} - q_{127}q_{236} \end{aligned} \tag{28}$$
$$\begin{aligned} f_4 := {} & -q_{125}q_{136}q_{237} - q_{125}q_{137}q_{236} + q_{126}q_{135}q_{237} \\ & - q_{126}q_{137}q_{235} - q_{127}q_{135}q_{236} - q_{127}q_{136}q_{235} + q_{127} - q_{136} + q_{235} \end{aligned} \tag{29}$$
$$\begin{aligned} f_5 := {} & -q_{124}q_{136}q_{237} + q_{124}q_{137}q_{236} - q_{126}q_{134}q_{237} \\ & + q_{126}q_{137}q_{234} + q_{127}q_{134}q_{236} + q_{127}q_{136}q_{234} - q_{126} - q_{137} - q_{234} \end{aligned} \tag{30}$$
$$\begin{aligned} f_6 := {} & q_{124}q_{135}q_{237} - q_{124}q_{137}q_{235} + q_{125}q_{134}q_{237} \\ & - q_{125}q_{137}q_{234} - q_{127}q_{134}q_{235} - q_{127}q_{135}q_{234} + q_{125} + q_{134} - q_{237} \end{aligned} \tag{31}$$
$$\begin{aligned} f_7 := {} & -q_{124}q_{135}q_{236} + q_{124}q_{136}q_{235} - q_{125}q_{134}q_{236} \\ & + q_{125}q_{136}q_{234} + q_{126}q_{134}q_{235} + q_{126}q_{135}q_{234} - q_{124} + q_{135} + q_{236}. \end{aligned} \tag{32}$$

5 Two SL_2 actions

A 2-dimensional maximal torus T of G_2 is described by Springer and Veldkamp in Section 2.3 of [21] as the subgroup of automorphisms of $\mathbb{O}$ consisting of the following transformations:

$$t_{\lambda,\mu}: \ (x, y) \mapsto (c_\lambda x c_\lambda^{-1}, c_\mu y c_\lambda^{-1}),$$

where $(x, y) \in \mathbb{O}$, $\lambda, \mu \in k^*$ and c_λ, c_μ are the diagonal matrices $\mathrm{diag}(\lambda, \lambda^{-1})$, $\mathrm{diag}(\mu, \mu^{-1})$, respectively.

We look more closely at how T acts on the grassmannian, so we express the action in our coordinates. Let e_{ij} denote the elementary 2×2 matrix which has 1 at i, j'th position and 0's elsewhere. The set of pairs $\{(e_{11}, 0), (e_{12}, 0), (e_{21}, 0), (e_{22}, 0), (0, e_{11}), (0, e_{12}), (0, e_{21}), (0, e_{22})\}$ forms a basis for $\mathbb{O}$. In this basis, $t_{\lambda,\mu}$ is the diagonal matrix

$$t_{\lambda,\mu} = \mathrm{diag}(1, \lambda^2, \lambda^{-2}, 1, \lambda^{-1}\mu, \lambda\mu, \lambda^{-1}\mu^{-1}, \lambda\mu^{-1}).$$

We are going to switch to the basis $\{e = (e_{11} + e_{22}, 0), e_1 = (\mathbf{i}, 0), e_2 = (\mathbf{j}, 0), e_3 = (\mathbf{k}, 0), e_4 = (0, e_{11} + e_{22}), e_5 = (0, \mathbf{i}), e_6 = (0, \mathbf{j}), e_7 = (0, \mathbf{k})\}$. The proof of the next lemma is straightforward so we skip it.

Lemma 5.1. The action of maximal torus $T = t_{\lambda,\mu}$ of G_2 on the basis $\{e_1, \ldots, e_7\}$ of $\mathbb{I}$ is given by

$$\begin{aligned}
t_{\lambda,\mu}(e) &= e \\
t_{\lambda,\mu}(e_1) &= e_1 \\
t_{\lambda,\mu}(e_2) &= \left(\frac{\lambda^2 + \lambda^{-2}}{2}\right) e_2 + i\left(\frac{-\lambda^2 + \lambda^{-2}}{2}\right) e_3 \\
t_{\lambda,\mu}(e_3) &= i\left(\frac{\lambda^2 - \lambda^{-2}}{2}\right) e_2 + \left(\frac{\lambda^2 + \lambda^{-2}}{2}\right) e_3 \\
t_{\lambda,\mu}(e_4) &= \left(\frac{\lambda^{-1}\mu + \lambda\mu^{-1}}{2}\right) e_4 + i\left(\frac{-\lambda^{-1}\mu + \lambda\mu^{-1}}{2}\right) e_5 \\
t_{\lambda,\mu}(e_5) &= i\left(\frac{\lambda^{-1}\mu - \lambda\mu^{-1}}{2}\right) e_4 + \left(\frac{\lambda^{-1}\mu + \lambda\mu^{-1}}{2}\right) e_5 \\
t_{\lambda,\mu}(e_6) &= \left(\frac{\lambda\mu + \lambda^{-1}\mu^{-1}}{2}\right) e_6 + i\left(\frac{-\lambda\mu + \lambda^{-1}\mu^{-1}}{2}\right) e_7 \\
t_{\lambda,\mu}(e_7) &= i\left(\frac{\lambda\mu - \lambda^{-1}\mu^{-1}}{2}\right) e_6 + \left(\frac{\lambda\mu + \lambda^{-1}\mu^{-1}}{2}\right) e_7.
\end{aligned}$$

There are two SL_2's naturally associated with the tori $t_{\lambda,\lambda}$ and $t_{\mathrm{id}_2,\mu}$.

Proposition 5.2. Let $x = (x_1, x_2)$ be an octonion from $\mathbb{I} = \mathfrak{sl}_2 \oplus \mathrm{Mat}_2$. The two SL_2 actions on $\mathbb{I}$ defined by

1. $g \cdot x = (g x_1 g^{-1}, g x_2 g^{-1})$ and

2. $g \cdot x = (x_1, g x_2)$

induce SL_2 actions on associative 3-planes.

Proof. Let $W \in \mathrm{Gr}(3,\mathbb{I})$ be an associative 3-plane spanned by the orthogonal basis $\{x,y,z\} \subset \mathbb{I} = \mathfrak{sl}_2 \oplus \mathrm{Mat}_2$. We know from Lemma 3.8 that W is associative if $[x,y,z]=0$. Thus, it suffices to check the vanishing of the associator

$$[g\cdot x, g\cdot y, g\cdot z] = (g\cdot x\, g\cdot y)g\cdot z - g\cdot x(g\cdot y\, g\cdot z).$$

Note that $\overline{g} = g^{-1}$ for all $g \in \mathrm{SL}_2$. Note also that for any $x=(x_1,x_2), y=(y_1,y_2)$ from $\mathbb{I} = \mathfrak{sl}_2 \oplus \mathrm{Mat}_2$ we have

$$\begin{aligned}(g\cdot x)(g\cdot y) &= (gx_1y_1g^{-1} + \overline{gy_2g^{-1}}gx_2g^{-1}, gy_2x_1g^{-1} + gx_2g^{-1}\overline{gy_1g^{-1}})\\ &= (gx_1y_1g^{-1} + g\overline{y_2}g^{-1}gx_2g^{-1}, gy_2x_1g^{-1} + gx_2g^{-1}g\overline{y_1}g^{-1})\\ &= (gx_1y_1g^{-1} + g\overline{y_2}x_2g^{-1}, gy_2x_1g^{-1} + gx_2\overline{y_1}g^{-1})\\ &= g\cdot((x_1,x_2)(y_1,y_2)).\end{aligned}$$

Therefore, if $[x,y,z]=0$, then

$$\begin{aligned}[g\cdot x, g\cdot y, g\cdot z] = (g\cdot x\, g\cdot y)g\cdot z - g\cdot x(g\cdot y\, g\cdot z) &= (g\cdot(xy))g\cdot z - g\cdot x(g\cdot(yz))\\ &= g\cdot((xy)z) - g\cdot(x(yz))\\ &= g\cdot((xy)z - x(yz))\\ &= g\cdot[x,y,z]\\ &= 0.\end{aligned}$$

Next, we check our claim for the second action:

$$\begin{aligned}(g\cdot x)(g\cdot y) &= (x_1y_1 + \overline{gy_2}gx_2, gy_2x_1 + gx_2\overline{y_1})\\ &= (x_1y_1 + \overline{y_2}x_2, g(y_2x_1 + x_2\overline{y_1}))\\ &= g\cdot((x_1,x_2)(y_1,y_2)).\end{aligned}$$

The rest follows as in the previous case. □

As a consequence of Proposition 5.2 we obtain two SL_2 actions on the Zariski closure X_{min} of the set of all associative 3-planes in $\mathrm{Gr}(3,\mathbb{I})$. (Our choice of notation X_{min} for the Zariski closure in $\mathrm{Gr}(3,\mathbb{I})$ of associative 3-planes will be explained in the Appendix.)

Remark 5.3. We denote by U the following unipotent subgroup:

$$U = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} :\ u \in \mathbb{C} \right\} \subset \mathrm{SL}_2.$$

The matrices of the actions of a generic element $g_u := \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in U$ on the ordered basis $e_1,\dots,e_7$ of $\mathbb{I}$ are given by

1. $[g_u] = \begin{pmatrix} 1 & -iu & -u & 0 & 0 & 0 & 0 \\ iu & 1/2\,u^2+1 & -i/2u^2 & 0 & 0 & 0 & 0 \\ u & -i/2u^2 & 1-1/2\,u^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -iu & -u \\ 0 & 0 & 0 & 0 & iu & 1/2\,u^2+1 & -i/2u^2 \\ 0 & 0 & 0 & 0 & u & -i/2u^2 & 1-1/2\,u^2 \end{pmatrix}$

2. $[g_u] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & u/2 & -i/2u \\ 0 & 0 & 0 & 0 & 1 & -i/2u & -u/2 \\ 0 & 0 & 0 & -u/2 & i/2u & 1 & 0 \\ 0 & 0 & 0 & i/2u & u/2 & 0 & 1 \end{pmatrix}.$

For both of these actions of U on X_{min} the fixed point sets are positive dimensional. Indeed, the points $[e_{123}]$ and $[-e_{126} + ie_{127} + ie_{136} + e_{137}]$ of $\mathbb{P}(\bigwedge^3 \mathbb{I})$ lie on X_{min} (this can be verified by using eqs. (19)–(25)) and both of these points are fixed by the first action of U. By a result of Horrocks [15], we know that the fixed point set of a unipotent group acting on a connected complete variety is connected. Therefore, the fixed point set of U on X_{min} is positive dimensional for the first action. Similarly, the points $[e_{123}]$ and $[-e_{346} + ie_{347} - ie_{356} + e_{357}]$ of X_{min} are fixed by the second action of U, hence the fixed point set of this action is also positive dimensional.

6 Torus fixed points

Now we go back to analyzing fixed point set of the maximal torus $t_{\lambda,\mu}$ of G_2 on X_{min}. The action of $t_{\lambda,\mu}$ on the basis $\{e_1, \dots, e_7\}$ is computed in the previous section. The eigenvalues are $1, \frac{1}{\lambda^2}, \lambda^2, \frac{\lambda}{\mu}, \frac{\mu}{\lambda}, \frac{1}{\lambda\mu}, \lambda\mu$ and the respective eigenvectors are

$$\begin{aligned} \widetilde{e_1} &= e_1, \\ \widetilde{e_2} &= -ie_2 + e_3, \\ \widetilde{e_3} &= ie_2 + e_3, \\ \widetilde{e_4} &= -ie_4 + e_5, \\ \widetilde{e_5} &= ie_4 + e_5, \\ \widetilde{e_6} &= -ie_6 + e_7, \\ \widetilde{e_7} &= ie_6 + e_7. \end{aligned}$$

For i, j and k from $\{1, \dots, 7\}$ we write $\widetilde{e}_{ijk}$ for $\widetilde{e}_i \wedge \widetilde{e}_j \wedge \widetilde{e}_k$. Accordingly, we write $\widetilde{p}_{ijk}$ for the transformed Plücker coordinate functions so that

$$\widetilde{p}_{ijk}(\widetilde{e}_{rst}) = \begin{cases} 1 & \text{if } i = r,\ j = s,\ k = t; \\ 0 & \text{otherwise.} \end{cases}$$

Our defining equations (19)–(25) become:

$$\widetilde{f}_1 := \widetilde{p}_{247} + \widetilde{p}_{356} = 0 \tag{33}$$
$$\widetilde{f}_2 := 2\widetilde{p}_{147} + 2\widetilde{p}_{156} + \widetilde{p}_{245} + \widetilde{p}_{345} - \widetilde{p}_{267} - \widetilde{p}_{367} = 0 \tag{34}$$
$$\widetilde{f}_3 := 2\widetilde{p}_{147} + 2\widetilde{p}_{156} + \widetilde{p}_{245} - \widetilde{p}_{345} - \widetilde{p}_{267} + \widetilde{p}_{367} = 0 \tag{35}$$
$$\widetilde{f}_4 := 2\widetilde{p}_{127} - 2\widetilde{p}_{136} + \widetilde{p}_{234} + \widetilde{p}_{235} + \widetilde{p}_{467} + \widetilde{p}_{567} = 0 \tag{36}$$
$$\widetilde{f}_5 := -2\widetilde{p}_{127} - 2\widetilde{p}_{136} + \widetilde{p}_{234} - \widetilde{p}_{235} + \widetilde{p}_{467} - \widetilde{p}_{567} = 0 \tag{37}$$
$$\widetilde{f}_6 := 2\widetilde{p}_{124} - 2\widetilde{p}_{135} - \widetilde{p}_{236} - \widetilde{p}_{237} + \widetilde{p}_{456} + \widetilde{p}_{457} = 0 \tag{38}$$
$$\widetilde{f}_7 := 2\widetilde{p}_{124} + 2\widetilde{p}_{135} - \widetilde{p}_{236} + \widetilde{p}_{237} + \widetilde{p}_{456} - \widetilde{p}_{457} = 0. \tag{39}$$

It is easily verified that the following 35 vectors are eigenvectors for the action of $t_{\lambda,\mu}$ on $\bigwedge^3 \mathbb{I}$ (together with the eigenvalues indicated on the left column):

$\frac{1}{\lambda^3\mu}$	$\widetilde{e}_{126}$
$\lambda^3\mu$	$\widetilde{e}_{137}$
μ^2	$\widetilde{e}_{157}$
μ^{-2}	$\widetilde{e}_{146}$
$\frac{\lambda^2}{\mu^2}$	$\widetilde{e}_{346}$
$\frac{\mu^2}{\lambda^2}$	$\widetilde{e}_{257}$
$\frac{\lambda^3}{\mu}$	$\widetilde{e}_{134}$
$\frac{\mu}{\lambda^3}$	$\widetilde{e}_{125}$
$\frac{1}{\lambda^2\mu^2}$	$\widetilde{e}_{246}$
$\lambda^2\mu^2$	$\widetilde{e}_{357}$
λ^4	$\widetilde{e}_{347}$
λ^{-4}	$\widetilde{e}_{256}$
$\frac{\mu}{\lambda}$	$\widetilde{e}_{567},\ \widetilde{e}_{235},\ \widetilde{e}_{127}$
$\frac{\lambda}{\mu}$	$\widetilde{e}_{467},\ \widetilde{e}_{234},\ \widetilde{e}_{136}$
$\frac{1}{\lambda\mu}$	$\widetilde{e}_{456},\ \widetilde{e}_{236},\ \widetilde{e}_{124}$
$\lambda\mu$	$\widetilde{e}_{457},\ \widetilde{e}_{237},\ \widetilde{e}_{135}$
$\frac{1}{\lambda^2}$	$\widetilde{e}_{267},\ \widetilde{e}_{245},\ \widetilde{e}_{156}$
λ^2	$\widetilde{e}_{367},\ \widetilde{e}_{345},\ \widetilde{e}_{147}$
1	$\widetilde{e}_{356},\ \widetilde{e}_{247},\ \widetilde{e}_{167},\ \widetilde{e}_{145},\ \widetilde{e}_{123}$

Theorem 6.1. Among the eigenvectors of $t_{\lambda,\mu}$ in $\bigwedge^3 \mathbb{I}$, only the images of the following vectors in $\mathbb{P}(\bigwedge^3 \mathbb{I})$ lie in X_{min}:

$\lambda^2\mu^2$	$\widetilde{e}_{357}$
$\frac{1}{\lambda^2\mu^2}$	$\widetilde{e}_{246}$
$\frac{1}{\lambda^3\mu}$	$\widetilde{e}_{126}$
$\lambda^3\mu$	$\widetilde{e}_{137}$
μ^2	$\widetilde{e}_{157}$
$\frac{1}{\mu^2}$	$\widetilde{e}_{146}$
$\frac{1}{\lambda^4}$	$\widetilde{e}_{256}$
λ^4	$\widetilde{e}_{347}$
$\frac{\mu}{\lambda^3}$	$\widetilde{e}_{125}$
$\frac{\lambda^3}{\mu}$	$\widetilde{e}_{134}$
$\frac{\mu^2}{\lambda^2}$	$\widetilde{e}_{257}$
$\frac{\lambda^2}{\mu^2}$	$\widetilde{e}_{346}$
1	$\widetilde{e}_{167}$, $\widetilde{e}_{145}$, $\widetilde{e}_{123}$

Proof. It is easily checked that the points that are given in the hypothesis of the theorem are all torus fixed and all of them lie in X_{min}. The only place we have to be careful is that X_{min} may intersect eigenspaces of dimension ≥ 2. Nevertheless, this potential problem does not occur; when we substitute a nontrivial linear combination of eigenvectors belonging to the same eigenvalue into equations (33)–(39), we get a contradiction. □

7 Smoothness

Before proving our next result we recall the well-known "Jacobian Criterion for Smoothness."

Theorem 7.1. Let $I = (f_1, \dots, f_m)$ be an ideal from $\mathbb{C}[x_1, \dots, x_n]$ and let $x \in V(I)$ be a point from the vanishing locus of I in $\mathbb{C}^n$. Suppose $d = \dim V(I)$. If the rank of the Jacobian matrix $(\partial f_i/\partial x_j)_{i=1,\dots,m,\ j=1,\dots,n}$ at x is equal to $n - d$, then x is a smooth point of $V(I)$.

Theorem 7.2. The variety X_{min} is smooth.

Proof. If there is a singular point on X_{min}, then it has to occur at the torus fixed points. Thus, it suffices to analyze neighborhoods of fixed points by using affine charts that are described earlier.

We start with the fixed point $m = [\widetilde{e}_{123}]$, which lies on the open chart $\widetilde{U}_{123}$ as its origin. Here, "tilde" indicates that we are using transformed Plücker coordinates. Recall that X_{min} is cut-out on $\widetilde{U}_{123}$ by the vanishing of the seven linear forms (33)–(39). A straightforward calculation shows that the Jacobian of these polynomials with respect to variables $\widetilde{q}_{124}, \widetilde{q}_{125}, \widetilde{q}_{126}, \widetilde{q}_{127}, \widetilde{q}_{134}, \widetilde{q}_{135}, \widetilde{q}_{136}, \widetilde{q}_{137}, \widetilde{q}_{234}, \widetilde{q}_{235}, \widetilde{q}_{236}, \widetilde{q}_{237}$ (in the written order) evaluated at the origin (which is $\widetilde{e}_{123}$) is equal to

$$\mathrm{Jac}(\widetilde{f}_1,\dots,\widetilde{f}_7)|_{\widetilde{q}_{ijk}=0} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & -1/2 & 0 & 1/4 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & 0 & 0 & -1/2 & 0 & 1/4 & -1/4 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & -1/4 & -1/4 \\ 1/2 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & -1/4 & 1/4 \end{pmatrix}$$

which is obviously of rank 4. Hence, m is a smooth point of X_{min}.

We repeat this procedure for the other torus fixed points, which is tedious now. We verified this by using Maple. The outcome for each of the torus fixed points that are listed in Theorem 6.1 turns out to be the same. In summary, all of the 15 torus fixed points on X_{min} are nonsingular points, therefore, X_{min} is a smooth projective variety of dimension 8. □

8 Tangent space at $[\widetilde{e}_{123}]$

In this section we perform a sample calculation of the weights of a generic one-parameter $\gamma : \mathbb{C}^* \to t_{\lambda,\mu}$ subgroup on the tangent space of X_{min} at the $t_{\lambda,\mu}$-fixed point $m = [\widetilde{e}_{123}] \in X_{min}$. We use the term "generic" in the algebraic geometric sense, which is equivalent to the statement that the pairing between γ and any character $\alpha : t_{\lambda,\mu} \to \mathbb{C}^*$ is nonzero. In other words, we choose a regular one-parameter subgroup γ of $t_{\lambda,\mu}$ so that the fixed point set of γ on X_{min} is the same as that of $t_{\lambda,\mu}$. For example, $\gamma(s) := t_{s^{10},s}$, $s \in \mathbb{C}^*$ is regular.

Recall that the tangent space at $p = (p_1, \dots, p_n)$ of an affine variety $V \subseteq \mathbb{C}^n$ defined by the vanishing of the polynomials $f_1, \dots, f_r \in \mathbb{C}[x_1, \dots, x_n]$ is the intersection of the hyperplanes

$$\sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(p)(x_i - p_i) = 0, \quad \text{for } j = 1, \dots, r.$$

(Here we are abusing the notation. To be precise, x_i should be replaced by the vector field $\partial/\partial x_i$.) Equivalently, T_pV is the kernel of the Jacobian matrix of $f_1, \dots, f_r$ (with respect to x_i's) evaluated at the point $p \in V$. In our case, p is $m = [\widetilde{e}_{123}]$ (the origin of the tangent space) and the Jacobian with respect to local coordinates

$$\widetilde{q}_{124}, \widetilde{q}_{125}, \widetilde{q}_{126}, \widetilde{q}_{127}, \widetilde{q}_{134}, \widetilde{q}_{135}, \widetilde{q}_{136}, \widetilde{q}_{137}, \widetilde{q}_{234}, \widetilde{q}_{235}, \widetilde{q}_{236}, \widetilde{q}_{237}$$

is as given in the proof of Theorem 7.2. It is straightforward to verify that

$$\left\{ -\frac{1}{2}x_{135} + x_{237}, \frac{1}{2}x_{124} + x_{236}, -\frac{1}{2}x_{127} + x_{235}, \frac{1}{2}x_{136} + x_{234}, x_{137}, x_{134}, x_{126}, x_{125} \right\} \tag{40}$$

is a basis for the kernel of the Jacobian matrix computed in the proof of Theorem 7.2. Here, x_{ijk} stands for the tangent vector $\frac{\partial}{\partial \widetilde{q}_{ijk}}$.

Recall from Section 5 that $t_{\lambda,\mu}$ acts on $\mathbb{I}$ according to

$$\begin{aligned}
t_{\lambda,\mu}(\widetilde{e}_1) &= \widetilde{e}_1\\
t_{\lambda,\mu}(\widetilde{e}_2) &= \frac{1}{\lambda^2}\,\widetilde{e}_2\\
t_{\lambda,\mu}(\widetilde{e}_3) &= \lambda^2\,\widetilde{e}_3\\
t_{\lambda,\mu}(\widetilde{e}_4) &= \frac{\lambda}{\mu}\,\widetilde{e}_4\\
t_{\lambda,\mu}(\widetilde{e}_5) &= \frac{\mu}{\lambda}\,\widetilde{e}_5\\
t_{\lambda,\mu}(\widetilde{e}_6) &= \frac{1}{\lambda\mu}\,\widetilde{e}_6\\
t_{\lambda,\mu}(\widetilde{e}_7) &= \lambda\mu\,\widetilde{e}_7.
\end{aligned}$$

Let us denote by $w_{ijk}(\lambda,\mu)$ the weight (the eigenvalue) of the action $t_{\lambda,\mu}\cdot\widetilde{e}_{ijk}$.

The action of $t_{\lambda,\mu}$ on a Plücker coordinate $\widetilde{p}_{ijk}$ is given by

$$t_{\lambda,\mu}\cdot\widetilde{p}_{ijk}(x)=\widetilde{p}_{ijk}(t_{\lambda^{-1},\mu^{-1}}\cdot x)=w_{ijk}(\lambda^{-1},\mu^{-1})\widetilde{p}_{ijk},$$

and therefore, its action on a local Plücker coordinate function $\widetilde{q}_{rst}$ on $\widetilde{U}_{ijk}$ is given by

$$t_{\lambda,\mu}\cdot\widetilde{q}_{rst}(x)=\frac{\widetilde{p}_{rst}(t_{\lambda^{-1},\mu^{-1}}\cdot x)}{\widetilde{p}_{ijk}(t_{\lambda^{-1},\mu^{-1}}\cdot x)}=\frac{w_{rst}(\lambda^{-1},\mu^{-1})}{w_{ijk}(\lambda^{-1},\mu^{-1})}\widetilde{q}_{rst}.$$

Consequently, if $v=\sum_{r,s,t}a_{rst}\frac{\partial}{\partial\widetilde{q}_{rst}}$ is a tangent vector at $\widetilde{e}_{ijk}\in\widetilde{U}_{ijk}$, then the action of the one-parameter subgroup $\gamma(\lambda)=t_{\lambda^{10},\lambda}$, $\lambda\in\mathbb{C}^*$ on v is given by

$$\gamma\cdot v=\sum_{r,s,t}\lim_{\lambda\to 1}\left(\frac{w_{rst}(\lambda^{10},\lambda)}{w_{ijk}(\lambda^{10},\lambda)}\right)\frac{\partial}{\partial\widetilde{q}_{rst}}. \tag{41}$$

For example, the action of γ on the basis vectors (40), which we denote by $v_1,\ldots,v_8$ in the written order, is given by

$$\begin{aligned}
\gamma\cdot v_1 &= 11v_1\\
\gamma\cdot v_2 &= -11v_2\\
\gamma\cdot v_3 &= -9v_3\\
\gamma\cdot v_4 &= 9v_4\\
\gamma\cdot v_5 &= 31v_5\\
\gamma\cdot v_6 &= 29v_6\\
\gamma\cdot v_7 &= -31v_7\\
\gamma\cdot v_8 &= -29v_8.
\end{aligned}$$

9 Białynicki-Birula Decomposition

Let X be a smooth projective variety over $\mathbb{C}$ on which an algebraic torus T acts with finitely many fixed points. Let T' be a 1 dimensional subtorus with $X^{T'} = X^T$. For $p \in X^{T'}$, define the sets

$$C_p^+ = \{y \in X : \lim_{s \to 0} s \cdot y = p, \ s \in T'\}$$

and

$$C_p^- = \{y \in X : \lim_{s \to \infty} s \cdot y = p, \ s \in T'\},$$

called the plus and minus cells of p, respectively.

The following result is customarily called the Białynicki-Birula decomposition theorem in the literature.

Theorem 9.1 ([6]). If X, T and T' are as in the above paragraph, then

1. both of the sets C_p^+ and C_p^- are locally closed subvarieties in X, furthermore they are isomorphic to an affine space;
2. if T_pX is the tangent space of X at p, then C_p^+ (resp., C_p^-) is T'-equivariantly isomorphic to the subspace T_p^+X (resp., T_p^-X) of T_pX spanned by the positive (resp., negative) weight spaces of the action of T' on T_pX.

As a consequence of the BB-decomposition, there exists a filtration

$$X^{T'} = V_0 \subset V_1 \subset \cdots \subset V_n = X, \qquad n = \dim X,$$

of closed subsets such that for each $i = 1, \dots, n$, $V_i - V_{i-1}$ is the disjoint union of the plus (resp., minus) cells in X of (complex) dimension i. It follows that the odd-dimensional integral cohomology groups of X vanish, the even-dimensional integral cohomology groups of X are free, and the Poincaré polynomial $P_X(t) := \sum_{i=0}^{2n} \dim H^i(X; \mathbb{C}) t^i$ of X is given by

$$P_X(t) = \sum_{p \in X^{T'}} t^{2 \dim C_p^+} = \sum_{p \in X^{T'}} t^{2 \dim C_p^-}.$$

Now, let T' denote the 1 dimensional subtorus of $T = t_{\lambda,\mu}$ that is given by the image of the regular one-parameter subgroup $\gamma(\lambda) = t_{\lambda^{10},\lambda}$, $\lambda \in \mathbb{C}^*$. In the rest of this section, we are going to compute the weights of T' on the tangent spaces at the torus fixed points. We have already made a sample calculation of this sort in Section 8.

1. $p = [\widetilde{e}_{246}]$. An eigenbasis for tangent space at p is given by

$$\{-x_{234} + x_{467}, -1/2x_{124} + x_{456}, x_{346}, x_{245} + x_{267}, x_{256}, 1/2x_{124} + x_{236}, x_{146}, x_{126}\}$$

The weights (in the order of the eigenvectors) are $31, 11, 40, 2, -18, 11, 20, -9$.

2. $p = [\widetilde{e}_{157}]$. An eigenbasis for tangent space at p is given by

$$\{x_{125}, -1/2\,x_{127} + x_{567}, 1/2\,x_{135} + x_{457}, x_{357}, x_{257}, x_{167}, x_{145}, x_{137}\}$$

The weights (in the order of the eigenvectors) are $-31, -11, 9, 20, -20, -2, -2, 29$.

3. $p = [\widetilde{e}_{256}]$. An eigenbasis for tangent space at p is given by

$$\{-x_{235} + x_{567}, x_{236} + x_{456}, 1/2\,x_{156} + e_{267}, x_{257}, x_{246}, -1/2\,x_{156} + x_{245}, x_{126}, x_{125}\}$$

The weights (in the order of the eigenvectors) are $31, 29, 10, 22, 18, 10, 9, 11$.

4. $p = [\widetilde{e}_{126}]$. An eigenbasis for tangent space at p is given by

$$\{1/2\,x_{156} + x_{267}, x_{256}, x_{246}, 1/2\,x_{124} + x_{236}, x_{167}, x_{146}, x_{125}, x_{123}\}$$

The weights (in the order of the eigenvectors) are $11, -9, 9, 10, 31, 29, 2, 31$.

5. $p = [\widetilde{e}_{167}]$. An eigenbasis for tangent space at p is given by

$$\{-1/2\,x_{127} + x_{567}, 1/2\,x_{136} + x_{467}, -1/2\,x_{147} + x_{367}, 1/2\,x_{156} + x_{267}, x_{157}, x_{146}, x_{137}, x_{126}\}$$

The weights (in the order of the eigenvectors) are $-9, 9, 10, -10, 2, -2, 31, -31$.

6. $p = [\widetilde{e}_{145}]$. An eigenbasis for tangent space at p is given by

$$\{1/2\,x_{135} + x_{457}, -1/2\,x_{124} + x_{456}, 1/2\,x_{147} + x_{345}, -1/2\,x_{156} + x_{245}, x_{157}, x_{146}, x_{134}, x_{125}\}$$

The weights (in the order of the eigenvectors) are $11, -11, 10, -10, 2, -2, 29, -29$.

7. $p = [\widetilde{e}_{123}]$. An eigenbasis for tangent space at p is given by

$$\{-1/2\,x_{135} + x_{237}, 1/2\,x_{124} + x_{236}, -1/2\,x_{127} + x_{235}, 1/2\,x_{136} + x_{234}, x_{137}, x_{134}, x_{126}, x_{125}\}$$

The weights (in the order of the eigenvectors) are $11, -11, -9, 9, 31, 29, -31, -29$.

8. $p = [\widetilde{e}_{137}]$. An eigenbasis for tangent space at p is given by

$$\{-1/2\,x_{147} + x_{367}, x_{357}, x_{347}, -1/2\,x_{135} + x_{237}, x_{167}, x_{157}, x_{134}, x_{123}\}$$

The weights (in the order of the eigenvectors) are $-11, -9, 9, -10, -31, -29, -2, -31$.

9. $p = [\widetilde{e}_{125}]$. An eigenbasis for tangent space at p is given by

$$\{x_{257}, x_{256}, -1/2\,x_{156} + x_{245}, -1/2\,x_{127} + x_{235}, x_{157}, x_{145}, x_{126}, x_{123}\}$$

The weights (in the order of the eigenvectors) are $11, -11, 9, 10, 31, 29, -2, 29$.

10. $p = [\widetilde{e}_{257}]$. An eigenbasis for tangent space at p is given by

$$\{x_{157}, x_{125}, -1/2\, x_{127} + x_{567}, x_{237} + x_{457}, x_{357}, x_{245} + x_{267}, x_{256}, -1/2\, x_{127} + x_{235}\}$$

The weights (in the order of the eigenvectors) are $20, -11, 9, 29, 40, -2, -22, 9$.

11. $p = [\widetilde{e}_{357}]$. An eigenbasis for tangent space at p is given by

$$\{[x_{137}, -x_{235} + x_{567}, 1/2\, x_{135} + x_{457}, x_{345} + x_{367}, x_{347}, x_{257}, -1/2\, x_{135} + x_{237}, x_{157}\}$$

The weights (in the order of the eigenvectors) are $9, 31, -31, -11, -2, 18, -40, -11, -20$.

12. $p = [\widetilde{e}_{146}]$. An eigenbasis for tangent space at p is given by

$$\{x_{346}, x_{246}, x_{167}, x_{145}, x_{134}, x_{126}, 1/2\, x_{136} + x_{467}, -1/2\, x_{124} + x_{456}\}$$

The weights (in the order of the eigenvectors) are $20, -20, 2, 2, 31, -29, 11, -9$.

13. $p = [\widetilde{e}_{347}]$. An eigenbasis for tangent space at p is given by

$$\{-x_{234} + x_{467}, x_{237} + x_{457}, -1/2\, x_{147} + x_{367}, x_{357}, x_{346}, 1/2\, x_{147} + x_{345}, x_{137}, x_{134}\}$$

The weights (in the order of the eigenvectors) are $-31, -29, -10, -18, -22, -10, -9, -11$

14. $p = [\widetilde{e}_{134}]$. An eigenbasis for tangent space at p is given by

$$\{x_{146}, x_{145}, x_{137}, x_{123}, x_{347}, x_{346}, 1/2\, x_{147} + x_{345}, 1/2\, x_{136} + x_{234}\}$$

The weights (in the order of the eigenvectors) are $-31, -29, 2, -29, 11, -11, -9, -10$.

15. $p = [\widetilde{e}_{346}]$. An eigenbasis for tangent space at p is given by

$$\{1/2\, x_{136} + x_{234}, x_{146}, x_{134}, 1/2\, x_{136} + x_{467}, x_{236} + x_{456}, x_{345} + x_{367}, x_{347}, x_{246}\}$$

The weights (in the order of the eigenvectors) are $-9, -20, 11, -9, -29, 2, 22, -40$.

Theorem 9.2. The Poincaré polynomial of X_{min} is

$$P_X(t^{1/2}) = 1 + t + 2t^2 + 2t^3 + 3t^4 + 2t^5 + 2t^6 + t^7 + t^8.$$

Proof. The proof follows from the discussion at the beginning of this section and the computations made above. $\square$

10 Generalities on Degenerate Compactifications

Let G be a connected semisimple, simply connected algebraic group defined over k. A G-variety X is called a regular embedding if it is comprised of finitely many G-orbits such that

1) any G-orbit closure is smooth and equal to the transversal intersection of codimension one orbit closures that contain it;

2) for any $x \in X$, the normal space $T_x(X)/T_x(G \cdot x)$ contains a dense orbit of the isotropy group $G_x = \{g \in G : \ g \cdot x = x\}$.

Since there exists finitely many G-orbits, there exists a (generic) point $x \in X$ such that the G-orbit $G \cdot x \simeq G/G_x$ is open in X.

Now suppose we have an involutory automorphism $\sigma : G \to G$. The (minimal) symmetric variety associated with σ is the affine algebraic variety G/H, where H is the subgroup of elements fixed by σ. We denote by $\widetilde{H}$ the normalizer of H in G. According to De Concini and Procesi [12], there exists a unique smooth projective regular embedding with open G-orbit $G/\widetilde{H}$. Such an embedding is called a wonderful compactification. For an introduction to this fascinating and important branch of algebraic geometry, we recommend Brion's lecture notes [7] and Timashev's book [22].

An irreducible representation V with highest weight λ of G is said to be spherical if there exists a nonzero vector v fixed by H. In this case, there exists a unique point $[v]$ (the line in direction of v) in $\mathbb{P}(V)$ which is fixed by $\widetilde{H}$. We call the Zariski closure Z_λ of the G-orbit $G \cdot [v]$ in $\mathbb{P}(V)$ a special compactification of $G/\widetilde{H}$. Similar to wonderful compactifications (which are always special), these are the smallest possible G-equivariant compactifications of $G/\widetilde{H}$ (see [11]) and first considered by Vust in [23].

In general, special compactifications are not normal, let alone being smooth. A normalization for these embeddings are first defined in [11] and it goes as follows:

The map $G/\widetilde{H} \to \mathbb{P}(V_\lambda)$ given by $g\widetilde{H} \mapsto g[v]$ extends to a map $q : X \to Z_\lambda \subseteq \mathbb{P}(V_\lambda)$, where X is the wonderful compactification of $G/\widetilde{H}$. Let $\mathcal{L}_\lambda = q^*(\mathcal{O}_{\mathbb{P}(V_\lambda)}(1))$ denote the pull-back of the hyperplane bundle of $\mathbb{P}(V_\lambda)$ to X. The ring $A(\mathcal{L}_\lambda) := \bigoplus_{n \geq 0} H^0(X, \mathcal{L}_\lambda^n)$ is generated in degree 1 (see [10]), hence it can be identified with the projective coordinate ring of the image of X in $\mathbb{P}(H^0(X, \mathcal{L})^*)$. In other words, $A(\mathcal{L}_\lambda)$ is the coordinate ring of the image of $p : X \to \mathbb{P}(H^0(X, \mathcal{L})^*)$, which we denote by Y_λ. Since X is smooth, Y_λ is normal. Furthermore, the subring of $A(\mathcal{L}_\lambda)$ that is generated by V_λ^* is isomorphic to the projective coordinate ring of Z_λ. The corresponding natural projection map $r : Y_\lambda \to Z_\lambda$ satisfies $q = r \circ p$.

Following Maffei's work [18], we are going to describe the G-orbits in Z_λ. To this end, we start by explaining the relevant terminology. Let $T \subseteq G$ be a σ-stable maximal torus (which always exists) such that the dimension of the neutral component of $\{t \in T : \ \sigma(t) = t^{-1}\}$ is maximal. These choices give us the root system $\Phi(G, T)$ and the associated *restricted root system*, $\widetilde{\Phi} := \{\alpha - \sigma(\alpha) : \alpha \in \Phi\}$. Note that the restricted root system does not need to be irreducible. There is a Borel subgroup $B \subset G$ such that $B \cap \sigma(B)$ has minimal possible dimension. Corresponding to this choice, we have the set of simple roots Δ and the associated restricted simple roots $\widetilde{\Delta}$.

Let $\Phi_0 \subset \Phi$ denote the subset of roots that are fixed by σ, Φ_1 denote $\Phi - \Phi_0$. Let Δ_0 denote $\Delta \cap \Phi_0$ and let $\Delta_1 = \Delta - \Delta_0$. There exists an induced involution $\overline{\sigma}$ on Δ_1 such that $\sigma(\alpha) + \overline{\sigma}(\alpha)$ is in the span of Δ_0 and if $\beta = \overline{\sigma}(\alpha)$, then $\tilde{\beta} = \tilde{\alpha}$. Furthermore, the action of σ on the simple roots can be described explicitly by using $\overline{\sigma}$: if $\alpha \in \Delta_0$, then $\sigma(\alpha) = \alpha$, and if $\alpha \in \Delta_1$, then $\sigma(\alpha) = -w_{\Delta_0}\overline{\sigma}(\alpha)$. Here, w_{Δ_0} is the longest element of the parabolic subgroup W_{Δ_0} of the Weyl group W of $\Phi(G, T)$. The support of a weight λ is the set of simple roots $\alpha \in \Delta$ such that $\kappa(\alpha, \lambda) \neq 0$, where $\kappa(\cdot, \cdot)$ is the Killing form. We denote the support of λ by $\mathrm{supp}(\lambda)$.

Recall that a highest weight λ of G is called spherical (with respect to H) if its highest weight vector $v \in V_\lambda$ is fixed by H. Let us denote by Ω the set (semigroup) of all highest weights of G relative to T and denote by $\Omega^+ \subset \Omega$ the subsemigroup of all spherical highest weights (of course, T has to be a σ-stable maximal torus). A highest weight is called special if $\sigma(\lambda) = -\lambda$.

If I and J are subsets of $\widetilde{\Delta}$, then define

$$\Delta_I(J) := \bigcup \{\text{connected components of } \widetilde{\Delta} - J \text{ which intersect } I\}.$$

It is shown in [12] that the boundary of $G/\widetilde{H}$ in its wonderful compactification X is a union $\cup_{\tilde{\alpha} \in \widetilde{\Delta}} X_{\tilde{\alpha}}$ of irreducible G-stable divisors which are indexed by the simple roots of the restricted root system. These boundary divisors intersect each other transversally, and moreover, the closure of any G-orbit is of the form $X_J := \cap_{\alpha \in J} X_\alpha$ for some subset $J \subseteq \widetilde{\Delta}$. Thus, $X_J = \overline{O_J}$ where O_J is the G-orbit determined by the subset $J \subseteq \widetilde{\Delta}$. The unique closed G-orbit $X_{\widetilde{\Delta}}$ is isomorphic to G/P_{Δ_0}, where P_{Δ_0} is the parabolic subgroup of G corresponding to the set of (σ-fixed) simple roots Δ_0.

It is shown by Maffei in [18] that the G-orbits of the special compactification Z_λ of $G/\widetilde{H}$ are precisely the images of the G-orbits in X under $q : X \to Z_\lambda$ which is described above. Furthermore, for subsets K and J of $\widetilde{\Delta}$ we have

$$q(O_J) = q(O_K) \iff \Delta_I(J) = \Delta_I(K),$$

where $I \subseteq \widetilde{\Delta}$ is the support of λ in $\widetilde{\Delta}$.

We know from [12] that each subset $J \subseteq \widetilde{\Delta}$, the G-orbit O_J as well as its closure X_J is a fibration over a suitable partial flag variety. Indeed, let $d(J)$ denote Δ_0 union the set of restricted simple roots $\tilde{\alpha}$ whose restriction to the maximal σ-split torus of G lies in J. In this case, the parabolic subgroup $P_{d(J)}$ and $\sigma(P_{d(J)})$ are opposite parabolic subgroup, hence $L_J := P_{d(J)} \cap \sigma(P_{d(J)})$ is the Levi subgroup of $P_{d(J)}$. Let H_J denote the fixed subgroup of σ on L_J. Then

$$O_J \simeq G \times_{\sigma(P_{d(J)})} (L_J/H_J) \quad \text{and} \quad X_J \simeq G \times_{\sigma(P_{d(J)})} \overline{(L_J/H_J)},$$

where $\overline{(L_J/H_J)}$ is the wonderful compactification of L_J/H_J. In other words, the orbit O_J is a fibration over $G/P_{d(J)}$ and its fibers are isomorphic to the symmetric variety L_J/H_J, and similarly X_J is a fibration over $G/P_{d(J)}$ and its fibers are isomorphic to the wonderful compactification of the symmetric variety L_J/H_J.

11 The wonderful compactification of G_2/SO_4.

According to Borel and de Siebenthal [5], G_2 has a unique involution $\sigma : G_2 \to G_2$ whose fixed point subgroup H is one of its two maximal subgroups, namely SO_4. The standard maximal torus T of G_2 is split relative to the involution determined by SO_4. We choose the Borel subgroup appropriately so that the induced involution on $\Phi(G_2, T)$ is given by $\sigma(\alpha) = -\alpha$, sending a simple root to its negative. Therefore, $\widetilde{\Delta} = \{\tilde{\alpha} : \alpha \in \Delta\} = \{2\alpha_1, 2\alpha_2\}$, where α_1 (short) and α_2 (long) are the simple roots of (G_2, T). Thus, there are 4 G_2-orbits in the wonderful compactification of G_2/SO_4 and these orbits correspond to the subsets of $\widetilde{\Delta}$. The closed orbit is $G_2/P_\emptyset = G_2/B$, hence the flag variety of G_2. The other two orbits are fibrations over grassmannians $G_2/P_{\{\alpha_1\}}$ and $G_2/P_{\{\alpha_2\}}$.

Theorem 11.1 (De Concini - Springer [13], Example 3.4)**.** The Poincaré polynomial of the wonderful compactification X_{won} of G_2/SO_4 is

$$P_{X_{won}}(t^{1/2}) = 1 + 2t + 4t^2 + 4t^3 + 5t^4 + 4t^5 + 4t^6 + 2t^7 + t^8.$$

12 Minimal degenerate compactification of G_2/SO_4.

Recall from Section 2 that G_2/SO_4 embeds into $\mathbb{P}(\Gamma_{2,0})$, where $\Gamma_{2,0}$ is the irreducible representation of G_2 with the highest weight $2\omega_1$. Since ω_1 is the Killing dual of α_1, we see that $\text{supp}_{\widetilde{\Delta}}(2\omega_1) = \{2\alpha_1\}$. The set of restricted simple roots has 4 subsets $\emptyset, \{2\alpha_1\}, \{2\alpha_2\}$ and $\widetilde{\Delta}$ itself. Then

$$\begin{aligned} \Delta_I(\emptyset) &= \widetilde{\Delta}, \\ \Delta_I(\{2\alpha_1\}) &= \emptyset, \\ \Delta_I(\{2\alpha_2\}) &= \{2\alpha_1\}, \\ \Delta_I(\{2\alpha_1, 2\alpha_2\}) &= \emptyset. \end{aligned}$$

Therefore, we see that our special compactification of G_2/SO_4 in $\mathbb{P}(\Gamma_{2,0})$ has 3 G_2-orbits; the images of G_2/SO_4, $G_2/P_{\{2\alpha_2\}}$, and of G_2/B in $\mathbb{P}(\Gamma_{2,0})$, where $P_{\{2\alpha_2\}}$ is the parabolic subgroup of G_2 that corresponds to the weight $2\alpha_2$. It follows from these observations that the Picard number of this minimal degenerate compactification cannot be one.

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