Lecture 3: Finite Element Discontinuous Galerkin Schemes

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Outline of the Third Lecture

- Discontinuous Galerkin (DG) method for hyperbolic conservation laws
- DG method for second order convection-diffusion equations
- Superconvergence
- Problems with $\delta$-singularities
- Concluding remarks
We are interested in solving a hyperbolic conservation law

\[ u_t + f(u)_x = 0 \]

In 2D it is

\[ u_t + f(u)_x + g(u)_y = 0 \]

and in system cases \( u \) is a vector, and the Jacobian \( f'(u) \) is diagonalizable with real eigenvalues.
Several properties of the solutions to hyperbolic conservation laws.

- The solution $u$ may become discontinuous regardless of the smoothness of the initial conditions.

- Weak solutions are not unique. The unique, physically relevant entropy solution satisfies additional entropy inequalities

$$U(u)_t + F(u)_x \leq 0$$

in the distribution sense, where $U(u)$ is a convex scalar function of $u$ and the entropy flux $F(u)$ satisfies $F'(u) = U'(u)f'(u)$. 
To solve the hyperbolic conservation law:

$$u_t + f(u)_x = 0,$$  \hspace{1cm} (1)

we multiply the equation with a test function \(v\), integrate over a cell \(I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]\), and integrate by parts:

$$\int_{I_j} u_t v dx - \int_{I_j} f(u)v_x dx + f(u_{j+\frac{1}{2}})v_{j+\frac{1}{2}} - f(u_{j-\frac{1}{2}})v_{j-\frac{1}{2}} = 0$$
Now assume both the solution $u$ and the test function $v$ come from a finite dimensional approximation space $V_h$, which is usually taken as the space of piecewise polynomials of degree up to $k$:

$$V_h = \{ v : v|_{I_j} \in P^k(I_j), \ j = 1, \cdots, N \}$$

However, the boundary terms $f(u_{j+1/2}), v_{j+1/2}$ etc. are not well defined when $u$ and $v$ are in this space, as they are discontinuous at the cell interfaces.
From the conservation and stability (upwinding) considerations, we take

- A single valued monotone numerical flux to replace \( f(u_{j+\frac{1}{2}}) \):

  \[
  \hat{f}_{j+\frac{1}{2}} = \hat{f}(u_{j+\frac{1}{2}}, u_{j+\frac{1}{2}})
  \]

  where \( \hat{f}(u, u) = f(u) \) (consistency); \( \hat{f}(\uparrow, \downarrow) \) (monotonicity) and \( \hat{f} \) is Lipschitz continuous with respect to both arguments.

- Values from inside \( I_j \) for the test function \( v \)

  \[
  v_{j+\frac{1}{2}}^-, \quad v_{j-\frac{1}{2}}^+
  \]
Hence the DG scheme is: find $u \in V_h$ such that

$$\int_{I_j} u_t v \, dx - \int_{I_j} f(u) v_x \, dx + \hat{f}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0 \quad (2)$$

for all $v \in V_h$. 
Time discretization could be by the TVD Runge-Kutta method (Shu and Osher, JCP 88). For the semi-discrete scheme:

\[
\frac{du}{dt} = L(u)
\]

where \( L(u) \) is a discretization of the spatial operator, the third order TVD Runge-Kutta is simply:

\[
\begin{align*}
  u^{(1)} &= u^n + \Delta t L(u^n) \\
  u^{(2)} &= \frac{3}{4} u^n + \frac{1}{4} u^{(1)} + \frac{1}{4} \Delta t L(u^{(1)}) \\
  u^{n+1} &= \frac{1}{3} u^n + \frac{2}{3} u^{(2)} + \frac{2}{3} \Delta t L(u^{(2)})
\end{align*}
\]
Properties and advantages of the DG method:

- Easy handling of complicated geometry and boundary conditions (common to all finite element methods). Allowing hanging nodes in the mesh (unique to DG);

- Compact. Communication only with immediate neighbors, regardless of the order of the scheme;

- Explicit. Because of the discontinuous basis, the mass matrix is local to the cell, resulting in explicit time stepping (no systems to solve);

- Parallel efficiency. Achieves 99% parallel efficiency for static mesh and over 80% parallel efficiency for dynamic load balancing with adaptive meshes (Flaherty et al.).
- Provable cell entropy inequality and $L^2$ stability, for arbitrary nonlinear equations in any spatial dimension and any triangulation, for any polynomial degrees, without limiters or assumption on solution regularity (Jiang and Shu, Math. Comp. 94 (scalar case); Hou and Liu, JSC 07 (symmetric systems)). For $U(u) = \frac{u^2}{2}$:

$$\frac{d}{dt} \int_{I_j} U(u) dx + \hat{F}_{j+1/2} - \hat{F}_{j-1/2} \leq 0$$

Summing over $j$:

$$\frac{d}{dt} \int_a^b u^2 dx \leq 0.$$ 

This also holds for fully discrete RKDG methods with third order TVD Runge-Kutta time discretization, for linear equations (Zhang and Shu, SINUM 10).
Discontinuous Galerkin method

- At least $(k + \frac{1}{2})$-th order accurate, and often $(k + 1)$-th order accurate for smooth solutions when piecewise polynomials of degree $k$ are used, regardless of the structure of the meshes, for smooth solutions (Lesaint and Raviart 74; Johnson and Pitkäranta, Math. Comp. 86 (linear steady state); Zhang and Shu, SINUM 04 and 06 (RKDG for nonlinear equations)).

- $(2k + 1)$-th order superconvergence in negative norm and in strong $L^2$-norm for post-processed solution for linear and nonlinear equations with smooth solutions (Cockburn, Luskin, Shu and Süli, Math. Comp. 03; Ryan, Shu and Atkins, SISC 05; Curtis, Kirby, Ryan and Shu, SISC 07; Ji, Xu and Ryan, JSC 2013).
Discontinuous Galerkin method

- $(k + 3/2)$-th or $(k + 2)$-th order superconvergence of the DG solution to a special projection of the exact solution, and non-growth of the error in time up to $t = O\left(\frac{1}{\sqrt{h}}\right)$ or $t = O\left(\frac{1}{h}\right)$, for linear and nonlinear hyperbolic and convection diffusion equations (Cheng and Shu, JCP 08; Computers & Structures 09; SINUM 10; Meng, Shu, Zhang and Wu, SINUM 12 (nonlinear); Yang and Shu, SINUM 12 ($(k + 2)$-th order)).

- Easy $h$-$p$ adaptivity.

- Stable and convergent DG methods are now available for many nonlinear PDEs containing higher derivatives: convection diffusion equations, KdV equations, ...
Three examples

We show three examples to demonstrate the excellent performance of the DG method.

The first example is the linear convection equation

\[ u_t + u_x = 0, \quad \text{or} \quad u_t + u_x + u_y = 0, \]

on the domain \((0, 2\pi) \times (0, T)\) or \((0, 2\pi)^2 \times (0, T)\) with the characteristic function of the interval \((\frac{\pi}{2}, \frac{3\pi}{2})\) or the square \((\frac{\pi}{2}, \frac{3\pi}{2})^2\) as initial condition and periodic boundary conditions.
Figure 1: Transport equation: Comparison of the exact and the RKDG solutions at $T = 100\pi$ with second order ($P^1$, left) and seventh order ($P^6$, right) RKDG methods. One dimensional results with 40 cells, exact solution (solid line) and numerical solution (dashed line and symbols, one point per cell)
Figure 2: Transport equation: Comparison of the exact and the RKDG solutions at $T = 100\pi$ with second order ($P^1$, left) and seventh order ($P^6$, right) RKDG methods. Two dimensional results with $40 \times 40$ cells.
The second example is the double Mach reflection problem for the two dimensional compressible Euler equations.
Figure 3: Double Mach reflection. $\Delta x = \Delta y = \frac{1}{240}$. Top: $P^1$; bottom: $P^2$.  
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Figure 4: Double Mach reflection. Zoomed-in region. Top: $P^2$ with $\Delta x = \Delta y = \frac{1}{240}$; bottom: $P^1$ with $\Delta x = \Delta y = \frac{1}{480}$. 
Figure 5: Double Mach reflection. Zoomed-in region. $P^2$ elements. Top: $\Delta x = \Delta y = \frac{1}{240}$; bottom: $\Delta x = \Delta y = \frac{1}{480}$. 
The third example is the flow past a forward-facing step problem for the two dimensional compressible Euler equations. No special treatment is performed near the corner singularity.
Figure 6: Forward facing step. Zoomed-in region. $\Delta x = \Delta y = \frac{1}{320}$. Left: $P^1$ elements; right: $P^2$ elements.
Here is a (very incomplete) history of the early study of DG methods for convection dominated problems:

- 1973: First discontinuous Galerkin method for steady state linear scalar conservation laws (Reed and Hill).
- 1974: First error estimate (for tensor product mesh) of the discontinuous Galerkin method of Reed and Hill (Lesaint and Raviart).
- 1986: Error estimates for discontinuous Galerkin method of Reed and Hill (Johnson and Pitkäranta).

• 1997-1998: Discontinuous Galerkin method for convection diffusion problems (Bassi and Rebay, Cockburn and Shu, Baumann and Oden, ...).

• 2002: Discontinuous Galerkin method for partial differential equations with third or higher order spatial derivatives (KdV, biharmonic, ...) (Yan and Shu, Xu and Shu, ...)

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Collected works on the DG methods:


Discontinuous Galerkin Method


Difficulty in the generalization of DG

A naive generalization of the DG method to a PDE containing higher spatial derivatives could have disastrous results.

Consider, as a simple example, the heat equation

$$u_t - u_{xx} = 0$$  \hspace{1cm} (3)

for $x \in [0, 2\pi]$ with periodic boundary conditions and with an initial condition $u(x, 0) = \sin(x)$. 
A straightforward generalization of the DG method from the hyperbolic equation \( u_t + f(u)_x = 0 \) is to write down the same scheme and replace \( f(u) \) by \(-u_x\) everywhere: find \( u \in V_h \) such that, for all test functions \( v \in V_h \),

\[
\int_{I_j} u_t v dx + \int_{I_j} u_x v_x dx - \hat{u}_x j + \frac{1}{2} v^{-} j + \frac{1}{2} + \hat{u}_x j - \frac{1}{2} v^{+} j - \frac{1}{2} = 0
\]  

(4)

Lacking an upwinding consideration for the choice of the flux \( \hat{u}_x \) and considering that diffusion is isotropic, a natural choice for the flux could be the central flux

\[
\hat{u}_x j + \frac{1}{2} = \frac{1}{2} \left( (u_x)^- j + \frac{1}{2} + (u_x)^+ j + \frac{1}{2} \right)
\]

However the result is horrible!
Table 1: $L^2$ and $L^\infty$ errors and orders of accuracy for the “inconsistent” discontinuous Galerkin method (4) applied to the heat equation (3) with an initial condition $u(x, 0) = \sin(x)$, $t = 0.8$. Third order Runge-Kutta in time.

<table>
<thead>
<tr>
<th>$\Delta x$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$L^2$ error</td>
<td>order</td>
</tr>
<tr>
<td>$2\pi/20$</td>
<td>1.78E-01</td>
<td>—</td>
</tr>
<tr>
<td>$2\pi/40$</td>
<td>1.76E-01</td>
<td>0.016</td>
</tr>
<tr>
<td>$2\pi/80$</td>
<td>1.75E-01</td>
<td>0.004</td>
</tr>
<tr>
<td>$2\pi/160$</td>
<td>1.75E-01</td>
<td>0.001</td>
</tr>
</tbody>
</table>
Figure 7: The “inconsistent” discontinuous Galerkin method (4) applied to the heat equation (3) with an initial condition $u(x, 0) = \sin(x)$. $t = 0.8$. 160 cells. Third order Runge-Kutta in time. Solid line: the exact solution; Dashed line and squares symbols: the computed solution at the cell centers. Left: $k = 1$; Right: $k = 2$. 
It is proven in Zhang and Shu, *M3 AS 03*, that this "inconsistent" DG method for the heat equation is

- Consistent with the heat equation
- (very weakly) unstable
A good DG method for the heat equation: the local DG (LDG) method
(Bassi and Rebay, JCP 97; Cockburn and Shu, SINUM 98): rewrite the
heat equation as

\[ u_t - q_x = 0, \quad q - u_x = 0, \]  \hspace{2cm} (5)

and formally write out the DG scheme as: find \( u, q \in V_h \) such that, for all
test functions \( v, w \in V_h \),

\[
\int_{I_j} u_t v dx + \int_{I_j} q v x dx - \hat{q}_j + \frac{1}{2} v_{j+\frac{1}{2}} + \hat{q}_j - \frac{1}{2} v_{j-\frac{1}{2}} = 0
\]  \hspace{2cm} (6)

\[
\int_{I_j} q w dx + \int_{I_j} u w x dx - \hat{u}_j + \frac{1}{2} w_{j+\frac{1}{2}} + \hat{u}_j - \frac{1}{2} w_{j-\frac{1}{2}} = 0,
\]

\( q \) can be locally (within cell \( I_j \)) solved and eliminated, hence local DG.
A key ingredient of the design of the LDG method is the choice of the numerical fluxes \( \hat{u} \) and \( \hat{q} \) (remember: no upwinding principle for guidance).

The best choice for the numerical fluxes is the following alternating flux

\[
\hat{u}_{j+\frac{1}{2}} = u_{j+\frac{1}{2}}^-, \quad \hat{q}_{j+\frac{1}{2}} = q_{j+\frac{1}{2}}^+.
\]  

(7)

The other way around also works

\[
\hat{u}_{j+\frac{1}{2}} = u_{j+\frac{1}{2}}^+, \quad \hat{q}_{j+\frac{1}{2}} = q_{j+\frac{1}{2}}^-.
\]
We then have

- $L^2$ stability
- optimal convergence of $O(h^{k+1})$ in $L^2$ for $P^k$ elements

The conclusions are valid for general nonlinear multi-dimensional convection diffusion equations

$$u_t + \sum_{i=1}^{d} f_i(u) x_i - \sum_{i=1}^{d} \sum_{j=1}^{d} (a_{ij}(u) u_{x_j}) x_i = 0,$$

(8)

where $a_{ij}(u)$ are entries of a symmetric and semi-positive definite matrix, Cockburn and Shu, SINUM 98; Xu and Shu, CMAME 07 ($O(h^{k+1/2})$ or $O(h^k)$ in different cases).
Table 2: $L^2$ and $L^\infty$ errors and orders of accuracy for the LDG applied to the heat equation.

<table>
<thead>
<tr>
<th>$\Delta x$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
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<tbody>
<tr>
<td></td>
<td>$L^2$ error</td>
<td>order</td>
</tr>
<tr>
<td>$2\pi/20$, $u$</td>
<td>1.92E-03</td>
<td>—</td>
</tr>
<tr>
<td>$2\pi/20$, $q$</td>
<td>1.93E-03</td>
<td>—</td>
</tr>
<tr>
<td>$2\pi/40$, $u$</td>
<td>4.81E-04</td>
<td>2.00</td>
</tr>
<tr>
<td>$2\pi/40$, $q$</td>
<td>4.81E-04</td>
<td>2.00</td>
</tr>
<tr>
<td>$2\pi/80$, $u$</td>
<td>1.20E-04</td>
<td>2.00</td>
</tr>
<tr>
<td>$2\pi/80$, $q$</td>
<td>1.20E-04</td>
<td>2.00</td>
</tr>
<tr>
<td>$2\pi/160$, $u$</td>
<td>3.00E-05</td>
<td>2.00</td>
</tr>
<tr>
<td>$2\pi/160$, $q$</td>
<td>3.00E-05</td>
<td>2.00</td>
</tr>
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In recent years, there are a lot of efforts in the literature to obtain superconvergence results for DG methods solving hyperbolic conservation laws and other convection dominated PDEs.

Besides their own interests in numerical analysis, these superconvergence results are also useful for the design of asymptotically exact *a posteriori* error indicators, which are useful in adaptive computations.

There are mainly two categories of these superconvergence results.
The first category is to explore the superconvergence of the DG solution to the exact solution in negative norms.

The negative Sobolev norm is defined by

$$
\|v\|_{-k} = \max_{\varphi \in H^k, \varphi \neq 0} \frac{(v, \varphi)}{\|\varphi\|_{H^k}}
$$

where $(\cdot, \cdot)$ is the standard $L^2$ inner product and $H^k$ is standard Sobolev space of order $k$. 
For linear and certain nonlinear hyperbolic equations with smooth solutions, we have

$$\| u - u_h \|_{-k} \leq C h^{2k+1}$$  \hspace{1cm} (9)

where $C$ depends on $u$ and its derivatives but is independent of $h$ (Cockburn, Luskin, Shu and Süli, Math Comp 2003; Ji, Xu and Ryan, JSC 2013).

Even for linear hyperbolic equations with initial condition containing possible singularities including $\delta$-singularities, we still have the negative-norm error estimate

$$\| u - u_h \|_{-k-2} \leq C h^{k+1/2}$$

(Yang and Shu, Num Math to appear).
Together with similar superconvergence results for divided differences on uniform meshes, and with a local post-processing technique (Bramble and Schatz, Math Comp 1977), we can obtain a post-processed solution \( w_h = P(u_h) \) (where \( P \) is a local post-processing operator) on uniform meshes which is superconvergent in the strong \( L^2 \) norm:

\[
\| u - w_h \| \leq C h^{2k+1}
\]

(10)

where \( C \) depends on \( u \) and its derivatives but is independent of \( h \) (Cockburn, Luskin, Shu and Süli, Math Comp 2003).
These results have been generalized to one-sided post-processing near the boundaries (Ryan and Shu, MAA 2003), structured triangular meshes (Mirzaee, Ji, Ryan and Kirby, SINUM 2011), non-uniform meshes (Curtis, Kirby, Ryan and Shu, SISC 2007) and nonlinear problems (Ji, Xu and Ryan, JSC 2013). It has also been applied to aeroacoustics (Ryan, Shu and Atkins, SISC 2005) and computer graphics (Steffan, Curtis, Kirby and Ryan, IEEE-TVCG 2008).
Superconvergence towards a special projection or at special points

The second category of superconvergence is \((k + 3/2)\)-th or \((k + 2)\)-th order superconvergence of the DG solution to a special projection of the exact solution.

A typical result is of the form

\[
\|Pu - u_h\| \leq C(1 + t)h^{k+1+r}
\]

where \(Pu\) is the Gauss-Radau projection of the exact solution \(u\), \(C\) is independent of time \(t\), and \(r = 1/2\) or 1.
This result implies that the standard $L^2$ error

$$\|u - u_h\| \leq C h^{k+1}$$

for $C$ independent of $t$, namely non-growth of the error in time, up to $t = O\left(\frac{1}{h^r}\right)$. 
- Cheng and Shu, JCP 2008, linear hyperbolic equation, $r = 1/2$, Fourier analysis;
- Cheng and Shu, Computers & Structures 2009, linear convection-diffusion equation, $r = 1/2$, Fourier analysis;
- Cheng and Shu, SINUM 2010, linear hyperbolic and convection-diffusion equation, $r = 1/2$, non-uniform meshes;
- Meng, Shu, Zhang and Wu, SINUM 2012, nonlinear hyperbolic equation, $r = 1/2$, non-uniform meshes;
- Yang and Shu, SINUM 2012, linear hyperbolic equation, $r = 1$, non-uniform meshes.
Related work:

- Adjerid et al. (CMAME 2002, 2006) proved superconvergence of the DG solutions at Radau points for ODEs and performed local analysis and numerical experiments for PDEs.

- Zhang and Shu (Computers & Fluids 2005) explicitly give the formulation of the DG solution in the case of $P^1$ (piecewise linear) for the linear convection equation. The leading error term is shown to be of a constant magnitude independent of the time $t$.

- Zhong and Shu (CMAME 2011) showed that the error between the DG numerical solution and the exact solution is $(k + 2)$-th order superconvergent at the downwind-biased Radau points and $(2k + 1)$-th order superconvergent at the downwind point in each cell (Fourier analysis, uniform meshes).
We develop and analyze DG methods for solving hyperbolic conservation laws

\[ u_t + f(u)_x = g(x, t), \quad (x, t) \in \mathbb{R} \times (0, T], \]

\[ u(x, 0) = u_0(x), \quad x \in \mathbb{R}, \]

where the initial condition \( u_0 \), or the source term \( g(x, t) \), or the solution \( u(x, t) \) contains \( \delta \)-singularities.
• Such problems appear often in applications and are difficult to approximate numerically, especially for finite difference schemes.

• Many numerical techniques rely on modifications with smooth kernels (mollification) and hence may severely smear such singularities, leading to large errors in the approximation.

In Yang and Shu, Num Math to appear and Yang, Wei and Shu, JCP submitted, we develop, analyze and apply DG methods for solve hyperbolic equations with $\delta$-singularities. The DG methods are based on weak formulations and can be designed directly to solve such problems without modifications, leading to very accurate results.
Linear equations with singular initial condition

We consider the linear model equation

$$u_t + \beta u_x = 0$$

$$u(x, 0) = u^0(x)$$

where $\beta$ is a constant, $u^0(x)$ has compact support, has a sole $\delta$-singularity at $x = 0$ and is sufficiently smooth everywhere else.

Even though the initial condition $u^0(x)$ is no longer in $L^2$, it does have an $L^2$-projection to the DG space $V_h$, which we use as the initial condition for the DG scheme. For problems involving $\delta$-singularities, negative-order norm estimates are more natural. We have the following theorem in Yang and Shu, Num Math to appear:
Theorem: By taking $\Omega_0 + 2supp(K_h^{2k+2,k+1}) \subset \subset \Omega_1 \subset \subset \Omega \setminus \mathcal{R}_T$, we have

$$\| u(T) - u_h(T) \|_{-(k+1)} \leq C h^k, \quad (12)$$

$$\| u(T) - u_h(T) \|_{-(k+2)} \leq C h^{k+1/2}, \quad (13)$$

$$\| u(T) - u_h(T) \|_{-(k+1),\Omega_1} \leq C h^{2k+1}, \quad (14)$$

$$\| u(T) - K_h^{2k+2,k+1} \ast u_h(T) \|_{\Omega_0} \leq C h^{2k+1}, \quad (15)$$

where the positive constant $C$ does not depend on $h$. Here the mesh is assumed to be uniform for (15) but can be regular and non-uniform for the other three inequalities.
Numerical example: We solve the following problem

\[ u_t + u_x = 0, \quad (x, t) \in [0, \pi] \times (0, 1], \]
\[ u(x, 0) = \sin(2x) + \delta(x - 0.5), \quad x \in [0, \pi], \]

(16)

with periodic boundary condition \( u(0, t) = u(\pi, t) \). Clearly, the exact solution is

\[ u(x, t) = \sin(2x - 2t) + \delta(x - t - 0.5). \]
Table 3: $L^2$-norm of the error between the numerical solution and the exact solution for equation (16) after post-processing in the region away from the singularity.

<table>
<thead>
<tr>
<th>$N$</th>
<th>d</th>
<th>$\mathcal{P}^1$ polynomial</th>
<th>$\mathcal{P}^2$ polynomial</th>
<th>$\mathcal{P}^3$ polynomial</th>
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<tbody>
<tr>
<td></td>
<td></td>
<td>error</td>
<td>order</td>
<td>error</td>
</tr>
<tr>
<td>200</td>
<td>0.2</td>
<td>6.88E-05</td>
<td>-</td>
<td>8.40e-07</td>
</tr>
<tr>
<td>300</td>
<td>0.2</td>
<td>1.41E-05</td>
<td>3.92</td>
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<tr>
<td>400</td>
<td>0.2</td>
<td>5.89E-06</td>
<td>3.02</td>
<td>1.98e-11</td>
</tr>
<tr>
<td>500</td>
<td>0.2</td>
<td>3.01E-06</td>
<td>3.01</td>
<td>6.13e-12</td>
</tr>
<tr>
<td>600</td>
<td>0.2</td>
<td>1.74E-06</td>
<td>3.00</td>
<td>2.37e-12</td>
</tr>
</tbody>
</table>
We consider the following two dimensional problem

\[ u_t + u_x + u_y = 0, \quad (x, y, t) \in [0, 2\pi] \times [0, 2\pi] \]

\[ u(x, 0) = \sin(x + y) + \delta(x + y - 2\pi), \quad (x, y) \in [0, 2\pi] \times [0, 2\pi], \quad (17) \]

with periodic boundary condition. Clearly, the exact solution is

\[ u(x, t) = \sin(x + y - 2t) + \delta(x + y - 2t) + \delta(x + y - 2t - 2\pi). \]

We use \( Q^k \) polynomial approximation spaces with \( k = 1 \) and 2.
Table 4: $L^2$-norm of the error between the numerical solution and the exact solution for equation (17) after post-processing in the region away from the singularity.

<table>
<thead>
<tr>
<th>$N$</th>
<th>d</th>
<th>$Q^1$ polynomial error</th>
<th>order</th>
<th>$Q^2$ polynomial error</th>
<th>order</th>
</tr>
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<tbody>
<tr>
<td>400</td>
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<td>-</td>
<td>3.23e-08</td>
<td>-</td>
</tr>
<tr>
<td>500</td>
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<td>1.24E-05</td>
<td>3.32</td>
<td>2.47e-10</td>
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<td>600</td>
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<tr>
<td>700</td>
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<td>5.11e-12</td>
<td>5.47</td>
</tr>
<tr>
<td>800</td>
<td>0.4</td>
<td>3.01E-06</td>
<td>3.02</td>
<td>2.53e-12</td>
<td>5.29</td>
</tr>
</tbody>
</table>
The theory generalizes to linear systems in a straightforward way. We solve the following linear system

\[ u_t - v_x = 0, \quad (x, t) \in [0, 2] \times (0, 0.4], \]
\[ v_t - u_x = 0, \quad (x, t) \in [0, 2] \times (0, 0.4], \]
\[ u(x, 0) = \delta(x - 1), \quad v(x, 0) = 0, \quad x \in [0, 2]. \]

(18)

Clearly, the exact solution (the Green’s function) is

\[ u(x, t) = \frac{1}{2} \delta(x - 1 - t) + \frac{1}{2} \delta(x - 1 + t), \]
\[ v(x, t) = \frac{1}{2} \delta(x - 1 + t) - \frac{1}{2} \delta(x - 1 - t). \]
Figure 8: Solutions of $u$ (left) and $v$ (right) for (18) at $t = 0.4$. 
Linear equations with singular source terms

We consider the linear model equation

\[ u_t(x, t) + Lu(x, t) = g(x, t), \quad (x, t) \in \Omega \times (0, \infty), \]
\[ u(x, 0) = 0, \quad x \in \Omega, \]

with \( L \) being a linear differential operator that does not involve time derivatives and \( g(x, t) \) is a singular source term, for example \( g(x, t) = \delta(x) \). The singular source term can be implemented in the DG scheme in a straightforward way, since it involves only the integrals of the singular source term with test functions in \( V_h \).

By using Duhamel’s principle, we can prove the following theorem (Yang and Shu, Num Math to appear):
Theorem: Denote
\[ \mathcal{R}_T = I_i \cup (T - C \log(1/h)h^{1/2}, T + C \log(1/h)h^{1/2}), \]
where \( I_i \) is the cell which contains the concentration of the \( \delta \)-singularity on the source term. Then we have the following estimates

\[
\| u(T) - u_h(T) \|_{-(k+1)} \leq Ch^k, \tag{19}
\]
\[
\| u(T) - u_h(T) \|_{-(k+2)} \leq Ch^{k+1/2}, \tag{20}
\]
\[
\| u - u_h \|_{-(k+1), \Omega_1} \leq Ch^{2k+1}, \tag{21}
\]
\[
\| u(T) - K_h^{2k+2,k+1} \ast u_h(T) \|_{\Omega_0} \leq Ch^{2k+1}, \tag{22}
\]

where \( \Omega_0 + 2\text{supp}(K_h^{2k+2,k+1}) \subset \subset \Omega_1 \subset \subset \mathbb{R} \setminus \mathcal{R}_T \). Here the mesh is assumed to be uniform for (22) but can be regular and non-uniform for the other three inequalities.
Numerical example: We solve the following problem

\[ \begin{align*}
    u_t + u_x &= \delta(x - \pi), & (x, t) \in [0, 2\pi] \times (0, 1], \\
    u(x, 0) &= \sin(x), & x \in [0, 2\pi], \\
    u(0, t) &= 0, & t \in (0, 1].
\end{align*} \tag{23} \]

Clearly, the exact solution is

\[ u(x, t) = \sin(x - t) + \chi[\pi, \pi+t], \]

where \( \chi[a,b] \) denotes the indicator function of the interval \([a, b]\).
Table 5: $L^2$-norm of the error between the numerical solution and the exact solution for equation (23) after post-processing in the region away from the singularity.

<table>
<thead>
<tr>
<th>$N$</th>
<th>d</th>
<th>$P^1$ polynomial</th>
<th>$P^2$ polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>error</td>
<td>order</td>
</tr>
<tr>
<td>401</td>
<td>0.2</td>
<td>1.74E-06</td>
<td>-</td>
</tr>
<tr>
<td>801</td>
<td>0.2</td>
<td>5.92E-09</td>
<td>8.22</td>
</tr>
<tr>
<td>1601</td>
<td>0.2</td>
<td>7.36E-10</td>
<td>3.03</td>
</tr>
<tr>
<td>3201</td>
<td>0.2</td>
<td>9.19E-11</td>
<td>3.01</td>
</tr>
<tr>
<td>6401</td>
<td>0.2</td>
<td>1.15E-11</td>
<td>3.01</td>
</tr>
</tbody>
</table>
Rendez-vous algorithm

Even though our theory is established only for linear equations, the DG algorithm can be easily implemented for nonlinear hyperbolic equations involving \( \delta \)-singularities.

In Canuto, Fagnani and Tilli, SIAM J Control and Optimization 2012, the following problem

\[
\rho_t + F_x = 0, \quad x \in [0, 1], t > 0,
\]

\[
\rho(0, t) = u_0(x), \quad t > 0,
\]

is studied. Here \( \rho \) is the density function, which is always positive.
The flux $F$ is given by

$$F(t, x) = v(t, x)\rho(t, x),$$

and the velocity $v$ is defined by

$$v(t, x) = \int_{\mathbb{R}^n} (y - x)\xi(y - x)\rho(t, y)dy,$$

where $\xi(x)$ is a positive function and supported on a ball centered at zero with radius $R$. Canuto et al. proved that when $t$ tends to infinity, the density function $\rho$ will converge to some $\delta$-singularities, and the distances between any of them cannot be less than $R$. Some computational results are shown in Canuto et al. based on a first order finite volume method.
We use the DG algorithm with the positivity-preserving limiter in Zhang and Shu JCP 2010, which can maintain positivity without affecting the high order accuracy, to both the one and two dimensional Rendez-vous algorithms, in Yang and Shu, Num Math to appear and Yang, Wei and Shu, JCP submitted.
Figure 9: Numerical density at $t = 1000$ with $N = 400$ when using $P^0$ (left) and $P^1$ (right) polynomials.
In 2D, the model is

\[
\begin{align*}
\rho_t + \text{div}(\mathbf{v}\rho) &= 0, \quad \mathbf{x} \in [-1, 1]^2, \quad t > 0, \\
\rho(\mathbf{x}, 0) &= \rho_0(\mathbf{x}), \quad t > 0,
\end{align*}
\]

(25)

where the velocity \( \mathbf{v} \) is defined by

\[
\mathbf{v}(\mathbf{x}, t) = \int_{B_R(\mathbf{x})} (\mathbf{y} - \mathbf{x})\rho(\mathbf{y}, t) d\mathbf{y}.
\]

In this example, we take \( R = 0.1 \) and

\[
\rho_0(\mathbf{x}) = \begin{cases} 
1 & r < 0.5, \\
0 & r > 0.5,
\end{cases}
\]

where \( r = ||\mathbf{x}|| \) is the Euclidean norm of \( \mathbf{x} \).
In Canuto et al., the authors demonstrated that the exact solution should be a single delta placed at the origin.

However, when we use rectangle meshes, we observe more than one delta singularity for $R$ sufficiently small. This is because the meshes are not invariant under rotation.
Figure 10: Numerical density $\rho$ with a rectangular $100 \times 100$ mesh using $P^0$ elements. $R = 0.08$ (left) and $R = 0.1$ (right).
To tackle this problem, we follow the same ideas in Cheng and Shu, JCP 2010; CiCP 2012, and construct a special equal-angle-zoned mesh. The structure of the mesh is given in figure 11. By using such a special mesh, the limit density is a single delta placed at the origin.
Figure 11: Left: Equal-angle-zoned mesh. Right: Numerical density $\rho$ for (25) at $t = 2000$ with $N = 200$ using $P^0$ elements.
Pressureless Euler equations

Another important system admitting $\delta$-singularities in its solutions is the pressureless Euler equation

$$w_t + f(w)_x = 0, \quad t > 0, \quad x \in \mathbb{R},$$

\[(26)\]

\[
\begin{pmatrix}
\rho \\
m
\end{pmatrix}
, \quad
f(w) =
\begin{pmatrix}
m \\
\rho u^2
\end{pmatrix}
,
\]

with $m = \rho u$, where $\rho$ is the density function and $u$ is the velocity.

It is quite difficult to obtain stable schemes for solve this system, especially for high order schemes.

A good property of this system is that the density is always positive and
the velocity satisfies a maximum-principle. Thus, in 1D, the convex set

\[
G = \left\{ \mathbf{w} = \begin{pmatrix} \rho \\ m \end{pmatrix} : \rho > 0, a \rho \leq m \leq b \rho \right\},
\]

where

\[
a = \min u_0(x), \quad b = \max u_0(x),
\]

with \(u_0\) being the initial velocity, is invariant. In Yang, Wei and Shu, JCP submitted, we adapt the techniques in Zhang and Shu, JCP 2010 to design a limiter to guarantee that our DG solution stays in set \(G\) without affecting high order accuracy. This is also generalized to 2D. Our scheme is thus very robust, stable and high order accurate for this pressureless Euler system.
We consider the following initial condition

\[
\rho(x, y, 0) = \frac{1}{100}, \quad (u, v)(x, y, 0) = \left(-\frac{1}{10} \cos \theta, -\frac{1}{10} \sin \theta\right), \quad (28)
\]

where \(\theta\) is the polar angle.

Since all the particles are moving towards the origin, the density function at \(t > 0\) should be a single delta at the origin.
Figure 12: Numerical density (left) and velocity field (right) at $t = 0.5$ for the initial condition (28).
We consider the following initial condition

\[ \rho(x, y, 0) = \frac{1}{10}, \quad (u, v)(x, y, 0) = \begin{cases} (-0.25, -0.25) & x > 0, y > 0, \\ (0.25, -0.25) & x < 0, y > 0, \\ (0.25, 0.25) & x < 0, y < 0, \\ (-0.25, 0.25) & x > 0, y < 0. \end{cases} \]  

(29)

Figure 13 shows the numerical density and velocity field at \( t = 0.5 \). From the figure, we can observe \( \delta \)-singularities located at the origin and the two axes.
Figure 13: Numerical density (left) and velocity field (right) at $t = 0.5$ for initial condition (29).
We consider the following initial condition

\[ \rho(x, y, 0) = \frac{1}{100}, \quad (u, v)(x, y, 0) = \begin{cases} \cos \theta, \sin \theta & r < 0.3, \\ \left(-\frac{1}{2} \cos \theta, -\frac{1}{2} \sin \theta\right) & r > 0.3, \end{cases} \]

where \( r = \sqrt{x^2 + y^2} \) and \( \theta \) is the polar angle.

Figure 14 shows the numerical density (contour plot) and velocity field at \( t = 0.5 \). From the figure, we can observe \( \delta \)-shocks located on a circle and vacuum inside.
Figure 14: Numerical density (left) and velocity field (right) at $t = 0.5$ for initial condition (30).
We consider the following initial condition

\[ \rho(x, y, 0) = 0.5, \quad (u, v)(x, y, 0) = \begin{cases} 
(0.3, 0.4) & x > 0, y > 0, \\
(-0.4, 0.3) & x < 0, y > 0, \\
(-0.3, -0.4) & x < 0, y < 0, \\
(0.4, -0.3) & x > 0, y < 0.
\end{cases} \]

(31)

Figure 15 shows the numerical density (contour plot) and velocity field with \( N = 50 \) at \( t = 0.4 \). From the figure, we can observe that the numerical solution approximates the vacuum quite well.
Figure 15: Numerical density (left) and velocity field (right) at $t = 0.4$ with $N = 50$ for initial condition (31).
Discontinuous Galerkin (DG) methods are very flexible to geometry, boundary condition and $h$-$p$ adaptivity.

Stable and accurate DG methods can be designed for a wide spectrum of PDEs including conservation laws and convection-diffusion equations.

Superconvergence results can be obtained for DG methods which can help for the design of error indicators to guide adaptive computation.

DG methods give good results for problems containing $\delta$-singularities.
DISCONTINUOUS GALERKIN METHOD

The End

THANK YOU!